All trees contain a large induced subgraph having all degrees 1 (mod k)

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Abstract. We prove that, for integers $n \geq 2$ and $k \geq 2$, every tree with n vertices contains an induced subgraph of order at least $2\lfloor (n + 2k - 3)/(2k - 1) \rfloor$ with all degrees congruent to 1 modulo k. This extends a result of Radcliffe and Scott, and answers a question of Caro, Krasikov and Roditty.

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§1. Introduction

An old result of Gallai (see [3], Problem 5.17) asserts that for every graph G there is a vertex partition $V(G) = V_1 \cup V_2$ such that the induced subgraphs $G[V_1]$ and $G[V_2]$ have all degrees even; it follows immediately that every graph of order n has an induced subgraph with all degrees even with order at least $n/2$. Given a graph G, it is natural to ask for the maximal order $f_2(G)$ of an induced subgraph of G with all degrees odd. It has been conjectured (see [1]) that there is a constant $c > 0$ such that every graph G without isolated vertices satisfies $f_2(G) \geq c|G|$. Let $f_2(n) = \min\{f_2(G) : |G| = n \text{ and } \delta(G) \geq 1\}$. Caro [1] proved $f_2(n) \geq c\sqrt{n}$, for $n \geq 2$, and Scott [6] proved that $f_2(n) \geq n/900 \log n$.

The conjecture has been proved for some special classes of graph (see [1], [6]). Caro, Krasikov and Roditty [2] proved a result for trees and conjectured a better bound. Radcliffe and Scott [4] proved the best possible bound,

$$f_2(T) \geq 2\left\lfloor \frac{|T|+1}{3} \right\rfloor,$$

for every tree T.

In this paper we consider trees but address the more general problem of determining $f_k(T)$, the maximal order of an induced subgraph of T with all degrees congruent to $1 \bmod k$. This problem was raised by Caro, Krasikov and Roditty [2], who proved that

$$f_k(T) \geq \frac{2(|T|-1)}{3k}$$

for every tree T, and conjectured that

$$f_k(T) \geq \frac{|T|+2k-4}{k-1}.$$

This conjecture is not correct however. Here we prove the following best possible bound.

Theorem 1. *For every tree T and every integer $k \geq 2$ there is a set $S \subset V(T)$ such that*

$$|S| \geq 2 \left\lfloor \frac{|T| + 2k - 3}{2k - 1} \right\rfloor$$

and $|\Gamma(x) \cap S| \equiv 1 (\bmod k)$ for every $x \in S$. This bound is best possible for all values of $|T|$.

We remark that, for $k = 2$, this is the result of Radcliffe and Scott [4] mentioned above; this theorem therefore generalizes that result. Further results concerning induced subgraphs mod k can be found in [5].

§2. Proof of Theorem 1

In this section we give a proof of Theorem 1. The result for $k = 2$ is proved in [4]; we may therefore assume that $k \geq 3$.

We begin by showing that the asserted bound is best possible. Let S_a be a star with $a + 1$ vertices (i.e. the central vertex has degree a), and let $C_{a,b}$ be the graph obtained by taking an S_{a-1} and an S_{b-1}, and joining their centres by an edge (thus $C_{a,b}$ is a rather short caterpillar with $a + b$ vertices). It is immediate to check that, for $a, b \leq k$, the graph $C_{a,b}$ is extremal for the theorem. Larger extremal examples can be obtained by taking $C_{a,b}$ (with $a, b \leq k$) together with any number of copies of $C_{k,k}$ and identifying one endvertex from each graph.

We turn now to the proof that the lower bound holds. Define

$$f(n) = 2 \left\lfloor \frac{n + 2k - 3}{2k - 1} \right\rfloor . \tag{1}$$

For a tree T, we say that $S \subset V(T)$ *has good degrees in* T if the subgraph of T induced by S has all degrees congruent to 1 mod k, and that S *is good in* T if S has good degrees in T and $|S| \geq f(|T|)$.

We use a similar approach to that used in [4]. We suppose that T is a minimal counterexample to the assertion of Theorem 1; it is readily checked that $\mathrm{diam}(T) \geq 4$. Let W_0 be the set of endvertices of T, let W_1 be the set of endvertices of $T \setminus W_0$ and let W_2 be the set of endvertices of $T \setminus (W_0 \cup W_1)$. For $i = 0, 1, 2$ and $v \in V(T)$, let $\Gamma_i(v) = \Gamma(v) \cap W_i$ and let $d_i(v) = |\Gamma_i(v)|$.

We begin with two lemmas giving general useful facts about f_k and f. The lemmas which follow tighten our grip on the structure of T until it is squeezed out of existence.

Lemma 2. *For positive integers n and $a_1, \ldots, a_n$, we have*

$$\sum_{i=1}^{n} f(a_i) \geq f\big((\sum_{i=1}^{n} a_i) - n + 1\big).$$

Proof. Straightforward calculation. □

Lemma 3. *For all $a > k$ we have $f_k(S_a) \geq f(|S_a) + k) + 2$. For $1 \leq a \leq k-1$ we have $f_k(S_a) = 2 \geq f(|S_a| + k)$. Also $f_k(S_k) = 2 = f(|S_k| + k - 1) = f(2k)$.*

Proof. Follows easily from $f_k(S_a) = k\lfloor (a-1)/k \rfloor + 2$ and (1), since $|S_a| = a + 1$. □

Lemma 4. *Suppose that $x \in W_2$ and set $a = d_0(x)$, $b = d_1(x)$ and $c = |\{v \in \Gamma_1(x) : d_0(v) = k\}|$. Then*

$$b(k-1) \leq a + c.$$

Moreover, if $b(k-1) = a + c$ then $d_0(v) \leq k$ for all $v \in \Gamma_1(x)$.

Proof. Suppose that $b(k-1) > a + c$. Write $\Gamma_0(x) = \{v_1, v_2, \ldots, v_a\}$ and $\Gamma_1(x) = \{w_1, w_2, \ldots, w_b\}$. Renumbering the w_i if necessary, we may suppose that $w_1, w_2, \ldots, w_c$ have $d_0(w_i) = k$. Let T_i be the component of $T \setminus x$ containing w_i $(i = 1, 2, \ldots, b)$ and let T' be the 'large' portion remaining. Simply by looking for a good subset S which does not contain x and using Lemmas 2 and 3, we see that

$$\begin{aligned}
f_k(T) &\geq f_k(T') + \sum_{i=1}^{b} f_k(T_i) \\
&= f_k(T') + c f_k(S_k) + \sum_{i=c+1}^{b} f_k(T_i) \\
&\geq f(|T'|) + c f(|S_k| + k - 1) + \sum_{i=c+1}^{b} f(|T_i| + k) \\
&\geq f\left(|T'| + 2ck + \left(\sum_{i=c+1}^{b} |T_i| \right) + (bk - c) - b \right) \\
&= f(|T| - a - 1 + b(k-1) - c),
\end{aligned}$$

since $|T| = |T'| + c(k+1) + \sum_{i=c+1}^{b} |T_i| + a + 1$. Since, by assumption, $b(k-1) - a - c - 1 \geq 0$ we have $f_k(T) \geq f(|T|)$, a contradiction. (Recall that T was supposed to be a minimal counterexample to the theorem.)

Furthermore, if we have the equality $b(k-1) = a + c$, then it must be that $d_0(w_i) \leq k-1$ for $i = c+1, \ldots, b$, for otherwise some T_i has $f_k(T_i) \geq f(|T_i|+k)+2$ (which again gives $f_k(T) \geq f(|T|)$). □

Lemma 5. *If $x \in W_2$ then $d_0(x) \leq k-1$. In fact if $y \in \Gamma_1(x)$ then*

$$d_0(y) < k \Rightarrow d_0(x) \leq k - 1$$

and

$$d_0(y) \geq k \Rightarrow d_0(x) \leq k - 2.$$

Proof. We begin by proving the first half of the assertion. Suppose on the contrary that $x \in W_2$, $y \in \Gamma_1(x)$, $d_0(x) \geq k$ and $d_0(y) < k$. Let A be any set of k vertices from $\Gamma_0(x)$ and let z be any element of $\Gamma_0(y)$. Set $V_0 = A \cup \Gamma_0(y)$, so $|V_0| \leq 2k - 1$. We can find a good subset S' in $T' = T \setminus V_0$; let

$$S = \begin{cases} S' \cup A & x \in S' \\ S' \cup \{z, y\} & x \notin S'. \end{cases}$$

Note that if $x \notin S'$ then also $y \notin S'$. Clearly S has good degrees in T. Furthermore, we have

$$|S| \geq |S'| + 2 \geq f(|T'|) + 2 = f(|T'| + 2k - 1) \geq f(|T|).$$

Thus S is good in T, which is a contradiction.

For the second half of the assertion, let us assume that $x \in W_2$, $y \in \Gamma_1(x)$, $d_0(x) > k-2$ and $d_0(y) \geq k$. We show that this leads to a contradiction.

Let A be any set of $(k-1)$ vertices from $\Gamma_0(x)$, let B be any set of k vertices from $\Gamma_0(y)$, and let z be any element of B. Let $V_0 = A \cup B$, so $|V_0| = 2k-1$, and let $T' = T \setminus V_0$. If S' is a good subset of T' then S is a good subset of T, where

$$S = \begin{cases} S' \cup B & y \in S' \\ S' \cup A \cup B \cup \{y\} & y \notin S', x \in S' \\ S' \cup \{y, z\} & x, y \notin S' \end{cases}.$$

This is a contradiction, and we are done. □

Lemma 6. *If $x \in W_2$ then $d_0(x) = k-2$ and $d_1(x) = 1$. Furthermore, $d_0(y) = k$, where y is the unique element of $\Gamma_1(x)$.*

Proof. Using the notation of Lemma 4, set $a = d_0(x)$, $b = d_1(x)$ and $c = |\{v \in \Gamma_1(x) : d_0(v) = k\}|$. It follows from Lemma 5 that $a \leq k-1$, and from Lemma 4 we have $b(k-1) \leq a + c$. If $a < k-2$ this inequality has no solutions (since $b > 0$ and $0 \leq c \leq b$). If $a = k-2$ then we get

$$(b-1)(k-1) \leq c-1, \tag{2}$$

while if $a = k-1$ then

$$(b-1)(k-1) \leq c. \tag{3}$$

The only solution of (2) is $b = c = 1$, which is what was claimed. In (3), however, for $a = k-1$, there are more possibilities. Let us first consider the general case when $b = 1$, and so $\Gamma_1(x) = \{y\}$, say. Suppose that $d_0(y) \neq k$, and so $c = 0$. Thus we have equality in (3), and it follows from Lemma 4 that $d_0(y) \leq k$. Therefore we have $b = 1$, $d_0(x) = k-1$ and $d_0(y) \leq k-1$. Set $V_0 = \Gamma_0(x) \cup \Gamma_0(y) \cup \{y\}$. Now $|V_0| \leq 2k-1$ and by the minimality of T we can find a good subset S' in the tree $T' = T \setminus V_0$. Let z be any element of $\Gamma_0(y)$ and set

$$S = \begin{cases} S' \cup y \cup \Gamma_0(x) & x \in S' \\ S' \cup \{y, z\} & \text{otherwise.} \end{cases}$$

S is good in T, which is a contradiction. Therefore we must have $d_0(y) = k$, as asserted.

If $b \neq 1$, the only possibility is the special case $a = b = c = 2$ and $k = 3$. Let y_1 and y_2 be the two elements of $\Gamma_1(x)$, pick z_1 and z_2 with $z_i \in \Gamma_0(y_i)$ and pick $w \in \Gamma_0(x)$. Set $V_0 = \Gamma_0(x) \cup \{y_1, y_2\} \cup \Gamma_0(y_1) \cup \Gamma_0(y_2)$, so $|V_0| = 10$. There is some set $S' \subset T \setminus V_0$ which is good in $T \setminus V_0$. Set

$$S = \begin{cases} S' \cup (V_0 \setminus \{w\}) & x \in S' \\ S' \cup \{y_1, z_1, y_2, z_2\} & x \notin S'. \end{cases}$$

Then S has good degrees in T and $|S| \geq |S'| + 4 \geq f(|T| - 10) + 4 = f(|T|)$. Thus S is good in T, which is a contradiction.

So far we have proved that if $a = k - 1$ then $b = 1$ and $d_0(y) = k$; but this contradicts Lemma 5. The only remaining possibility is that asserted in the lemma.

$\square$

We are now ready to finish the proof of Theorem 1. Lemma 6 has given us a great deal of information about the neighbourhood of any $x \in W_2$. Now let $x_0 x_1 x_2 \ldots x_m$ be a path in T of maximal length. Since $\text{diam}(T) \geq 4$ we know that $m \geq 4$ and $x_i \in W_i$, for $i = 0, 1, 2$.

We split the proof into cases according to whether $d_0(x_3) = 0$ or not.

If $d_0(x_3) = 0$ then we shall find a large good subset in each component of $T \setminus x_3$. These components consist of: some number, possibly zero, of stars (coming from elements of $\Gamma_1(x_3)$); at least one copy of $C_{k-1,k+1}$, one for each element of $\Gamma_2(x_3)$; and the rest of the tree, say T'. Let S' be a good subset of T'. From each star T'' we can pick a good subset of size at least $2|T''|/(2k-1)$ and from each caterpillar we can pick a good subset of size $k+2$. Because of the form of f we simply need to ensure that the good subsets we find have total size at least $2|T \setminus T'|/(2k-1)$. This is clearly achieved in the stars, and more than achieved in the caterpillars, with enough spare to account to x_3. Thus the union of S' with these smaller good subsets is a good subset of T.

If $d_0(x_3) > 0$ then we use a slightly different approach. Let v be an element of $\Gamma_0(x_3)$ and consider $V_0 = \Gamma_0(x_1) \cup \Gamma_0(x_2) \cup \{v\}$. Note that $|V_0| = 2k - 1$. Let S' be a good subset of $T' = T \setminus V_0$ and let

$$S = \begin{cases} S' \cup \Gamma_0(x_1) & x_1, x_2 \in S' \\ S' \triangle \{v, x_0, x_1, x_2\} & x_2 \in S', x_1 \notin S' \\ S' \cup \{x_0, x_1\} & x_1, x_2 \notin S' \end{cases}.$$

Then S has good degrees in T and

$$|S| \geq |S'| + 2 \geq f(|T'|) + 2|V_0|/(2k - 1) = f(|T|).$$

Therefore S is good in T, which contradicts the claim that T is a counterexample to the theorem. We have therefore proved Theorem 1. □

The problem of determining for a tree T the largest $S \subset V(T)$ such that $T[S]$ has all degrees congruent to 0 modulo k is equivalent to the problem of determining the largest independent set. It would, however, be interesting to give bounds on the size of the largest $S \subset V(T)$ such that all vertices in $S[T]$ have either degree 1 or degree congruent to 0 modulo k.

In general, for graphs with minimal degree sufficiently large, it would also make sense to ask for bounds on the size of the largest induced subgraph with all degrees congruent to i modulo k, where $0 \leq i \leq k$.

References

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