RESEARCH STATEMENT

Abigail Raz (araz2@unl.edu)

My research interests are in discrete mathematics, which includes both combinatorics and graph theory. I am particularly interested in questions where probability may be introduced either by using probabilistic tools to tackle deterministic problems or taking classical deterministic results and examining their behavior in a random setting. I also hope to expand my work to combinatorial problems that can be tackled with other non-combinatorial tools, for example those with connections to statistical physics, theoretical computer science, and algebra. In this statement, I focus on a few recent results as well as some problems I am interested in investigating. The first result examines a hitting time problem in the random bipartite geometric graph. The second problem takes a classical graph theoretic result and now asks the same question of the Erdős-Rényi random graph. The final result diverges from the first two and provides a counter-example to an extension of the union-closed sets conjecture — an open problem in extremal combinatorics. After this I will move into a discussion of an undergraduate research project which I co-led and some additional possibilities for future work.

PERFECT MATCHINGS IN THE RANDOM BIPARTITE GEOMETRIC GRAPH

This is joint work with Xavier Pérez Giménez; a preliminary manuscript will be available soon.

Recall that a perfect matching in a graph G with n vertices is a set of $n/2$ disjoint edges. This project examines when perfect matchings first appear in the random bipartite geometric graph. A random bipartite geometric graph $G(n, n, r)$ is defined by randomly placing n red vertices and n blue vertices on a given metric space and adding edges (u, v), between vertices of different colors, sequentially by length ending with those where $d(u, v) = r$. In this project we focus on the d-dimensional unit cube and torus with the standard Euclidean distance.

It is a natural question to ask whether the first edge in the process that results in the minimum degree being at least 1 coincides, with high probability[1] with the first edge that creates a perfect matching. Similar results in other graph models are known beginning with the analogous question in Erdős-Rényi random graphs [6].

Result: In [10] we show that for the d-dimensional cube and d-dimensional torus, with $d \geq 3$, the first edge in the process that results in the minimum degree being at least one coincides, with high probability, with the first edge that creates a perfect matching.

It was previously known that the result does not hold in dimension 2. Our argument begins with a tool used by Shor and Yukich in [26]. We use this to identify the areas in the random graph where we may require long edges in our matching. In these regions we must proceed with more care, using an isoperimetric inequality of Brunn and Minkowski, see [21], to form local perfect matchings. We then carefully paste these local matchings together to form a perfect matching of the entire random graph.

Future Work: Going forward we intend to generalize our result to random geometric graphs on the d-dimensional cube using different metrics. It is likely to extend to those derived from l_p-norms, but may extend even more generally.

[1] With high probability ("w.h.p.") means with probability tending to 1 as $n \to \infty$

STRUCTURE OF THE LARGEST SUBGRAPHS OF $G_{n,p}$ WITH A GIVEN MATCHING NUMBER

Recall that the matching number of a graph G is the size of a largest set of disjoint edges and is denoted $\nu(G)$. We say the *size* of a graph is the number of edges. In what follows "largest" will refer to the size of the graph. Let us say a graph G has the *EG Property* if for each k every largest subgraph with matching number k has one of two forms:

(a) All edges are within a set of vertices of size $2k+1$.
(b) All edges are incident to a set of vertices of size k.

In 1959, Erdős and Gallai proved the following theorem in [5].

Theorem 1. *K_n has the EG Property.*

Erdős conjectured that this result can be extended from K_n to $\mathcal{K} = \binom{[n]}{l}$ for all l.

Conjecture 1. *(Erdős' Matching Conjecture) The largest subhypergraphs of $\mathcal{K} = \binom{[n]}{l}$ with matching number k have* $\max\left\{\binom{l(k+1)-1}{l}, \binom{n}{l} - \binom{n-k}{l}\right\}$ *hyperedges.*

The case $l = 2$ is Theorem 1. The conjecture has also been proved for $l = 3$ [7, 8, 15] and when k is not too close to n/l [7, 12]. Note that as k changes the optimal configuration shifts between two forms.

Result: In [23], I showed two regimes of p where Theorem 1 can be extended to $G_{n,p}$ (the usual Erdős-Rényi random graph — n vertices and each edge appears independently with probability $p = p(n)$) and one where it cannot.

Theorem 2. *If $p \geq \frac{8 \log n}{n}$ or $p \ll 1/n$, then with high probability $G_{n,p}$ has the EG Property. Furthermore, if $\beta/n < p < \frac{\gamma \log n}{n}$, where $\gamma < 1/3$ and $\beta > 4 \log 2$ then w.h.p. $G_{n,p}$ does not have the EG Property.*

Theorem 2 gives a good rough understanding of the ranges of p where we do or do not expect the EG-property. The most interesting part of the argument is for the upper range, where the proof uses the Tutte-Berge formula together with various probabilistic arguments and tools. In the middle range we show that the EG property fails at $k = \nu(G_{n,p})$.

Future Work: Going forward, one could attempt to close the gaps for p in Theorem 2. However, I am more interested in better understanding the range where $G_{n,p}$ does not have the EG property. Specifically, what happens for k other than (and maybe not too close to) $\nu(G_{n,p})$, since the negative part of the theorem considers only $k = \nu(G_{n,p})$. It would also be interesting to see what, if anything, can be said about the forms of the largest subgraphs with matching number k when $G_{n,p}$ fails to have the EG-property. Finally, another interesting line of study would be to see if this result extends to the case of random hypergraphs. Specifically, is it possible to prove a result in the case of random hypergraphs, for particular ranges of p, even when the analogous result in the deterministic setting is unknown?

A COUNTER-EXAMPLE TO AN EXTENSION OF THE UNION-CLOSED SETS CONJECTURE

We say a family of sets, $\mathcal{A}$, is union-closed if for all $A, B \in \mathcal{A}$ we have $A \cup B \in \mathcal{A}$. The union-closed sets conjecture states that if a finite family of sets $\mathcal{A} \neq \{\emptyset\}$ is union-closed, then there is an element which belongs to at least half the sets in $\mathcal{A}$. In 2001, Reimer showed in [24] that the average set size of a union-closed family, $\mathcal{A}$, is at least $\frac{1}{2}\log_2 |\mathcal{A}|$. In order to do so, he showed that all union-closed families satisfy a particular condition (that we will refer to as "Reimer's condition"), which in turn implies the preceding bound. The question of whether Reimer's condition alone is enough to imply that there is an element in at least half of the sets was raised in the context of Gowers' polymath project on the union-closed sets conjecture [11].

Result: I exhibited a counter-example with ground set $\{1, \ldots, 8\}$ that satisfies Reimer's condition, but fails to have an element in at least half the sets. The counter-example is minimal (both in the size of the ground set and the number of sets in the family) and can be found in [22]. I generated the counter-example by constructing an auxiliary directed graph and examining how changes to the potential counter-example affected the degrees of the auxiliary digraph. Using this one can show that a counter-example exists and easily recover it.

Future Work: It is important to note that this counter-example is very far from union-closed. In the future, it would be interesting to investigate whether additional counter-examples that are closer to being union-closed can be obtained via similar methods. I believe this line of inquiry would be well suited to an undergraduate student, particularly as this could be attacked with combinatorial machinery combined with clever computer searches. More broadly, while the conjecture itself is likely too challenging for an undergraduate project, it has produced many interesting variants and related lines of inquiry that are more accessible, see [3, 11].

UNDERGRADUATE RESEARCH PROJECTS

During this past summer I served as a co-mentor for two projects in the Polymath REU. Both groups focused on analyzing a particular game on graphs. I mentored four students from this REU through the process of speaking at the Young Mathematicians Conference. Below I discuss one of the problems that we investigated.

THE EXPLORER-DIRECTOR GAME

This game was first introduced by Nedev and Muthukrishnan in [19]. Two friends (Explorer and Director) are playing a game moving a token around on the vertices of a graph. Each turn, Explorer picks a distance that the token will move, and then Director moves the token to any vertex that is the specified distance away. Explorer is trying maximize the number of distinct visited vertices, and Director is trying to minimize this number. We call $f_d(G, v)$ the number of vertices visited if both players play optimally on graph G starting a vertex v.

Previously this game was only explored in the context of cycle graphs. In [20], our group was able to prove exact bounds for particular families of graphs such as trees and lattices. We were also able to provide upper and lower bounds for arbitrary graphs. Finally, we explored variants of the game, such as one where we allow the Director to move the token along any path of the given length rather than only a minimal path that length.

Future Work: This particular game still has many open avenues of work; there are still many natural graph families where $f_d(G, v)$ is unknown. One other possible line of investigation would be to consider how $f_d(G, v)$ is affected by different graph products. For students who are more interested in computer science and algorithms an interesting question would be to examine how quickly the Explorer can reach $f_d(G, v)$ vertices. This has been investigated by various authors in the case of cycles, see [4, 13, 17, 18], but not for other graph families. Additionally, for the more algebraically inclined student there is a natural generalization to groups that has similarly open questions, see [2, 9].

CONTINUING AND FUTURE WORK

In the previous section I noted a few directions for further research connected to my results. Here I discuss a few questions in another area of study, percolation theory, which is also of interest.

In the standard model of percolation theory, we consider the d-dimensional integer lattice (the graph consisting of the vertex set $\mathbb{Z}^d$ together with edges between any two points with Euclidean distance 1). Percolation theory, generally, examines the behavior of the random subgraph of $\mathbb{Z}^d$ where each edge is, independently, "open" with probability p and "closed" with probability $1-p$. A standard first question is "What is the probability that the origin can reach infinitely many vertices in our random subgraph?".

The first question pertains to percolation on any locally finite graph G (of which $\mathbb{Z}^d$ is one example). Again we say each edge is independently, "open" with probability p and "closed" with probability $1-p$, and let H denote the resulting random subgraph. The standard critical probability for percolation on a graph G is defined by

$$p_c(G) = \sup\{p : \mathbb{P}(H \text{ contains an infinite component}) = 0\}.$$

A natural question, first introduced by Lyons in [16], is to investigate if the first moment bound correctly identifies $p_c(G)$. In 2003, Kahn showed this to be false [14]. However, in the same paper he gave a natural way to try and salvage the original conjecture. Recently, I have begun working on Kahn's conjecture with two colleagues, Jinyoung Park and Corrine Yap. We presently suspect that even the weaker conjecture may be false, perhaps even for large families of graphs.

The last two questions represent the fascinatingly non-intuitive nature of many problems in percolation theory. The first question concerns only a finite $n \times m$ subset of $\mathbb{Z}^2$ and instead of choosing edges to be open or closed, we now assign them directions (thus producing a random directed graph).

Question 1. *Take a $n \times m$ grid. We randomly assign each edge a direction. The vertical edges will be assigned "up" with probability $\frac{1}{2}$ (and "down" also with probability $\frac{1}{2}$). The horizontal edges will be assigned "right" with probability p (and "left" with probability $1-p$). Let E be the event that there is a directed path from the left side to the right side. Is $\mathbb{P}(E)$ monotone with p?*

This question was told to me by Bhargav Narayanan and seems obviously true. Yet, to my knowledge, it remains open. If this statement is true a next step would be to more precisely describe the behavior of $\mathbb{P}(E)$.

The second question was told to me by Peter Winkler and does not seem to have an obvious "correct" answer. Admittedly in thinking about it I have changed my opinion (and ultimately reach no consensus) on what I believe to be the correct answer.

Question 2. *Let G be a subgraph of the lattice. If you start at the origin of the lattice and take one step "up" followed by a step "right" can you guarantee that there is some continuation of this sequence that will return to the origin?*

Percolation theory contains many similar questions, and, while some have simple answers, many more remain open.

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