Nonrobustness of Closed-Loop Stability for Infinite-Dimensional Systems Under Sample and Hold

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***Abstract*—It is a well-known principle, for finite-dimensional systems, that applying sampled-and-hold in the feedback loop around a stabilizing state feedback (or dynamic) controller results in a stable sampled-data feedback control system if the sampling period is small enough. The principle extends to infinite-dimensional systems with compact state feedback if either the input operator is bounded or the given state-space system is analytic. In this note, we give an example for which this principle fails but which nevertheless satisfies certain necessary conditions arising in sampled-data control of infinite-dimensional systems.**

***Index Terms*—Infinite-dimensional systems, nonrobustness, sampled-data control.**

I. Introduction

In this note, we consider a fundamental problem which arises in a context of sampled-data stabilization of infinite-dimensional control systems. Consider a system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = x^0. \tag{1.1}$$

Suppose that a continuous-time feedback control

$$u(t) = Fx(t) \tag{1.2}$$

is exponentially stabilizing for (1.1). This continuous-time control is implemented naturally using sample and hold, so that u is given instead by

$$u(t) = Fx(k\tau), \quad t \in [k\tau, (k+1)\tau) \quad k \in \mathbb{N}_0. \tag{1.3}$$

Here, $\tau > 0$ is the sampling period. The control (1.3) is called a sampled-data feedback control and the overall system given by (1.1) and (1.3) is referred to as a sampled-data feedback system. Intuitively, we would expect for all sufficiently small $\tau > 0$, that (1.3) produces a stabilizing control for (1.1) in the sense that there exists $N \geq 1$ and $\nu > 0$ so that

$$\|x(t)\| \leq Ne^{-\nu t}\|x^0\|, \quad \forall\, x^0, \quad \forall\, t \geq 0. \tag{1.4}$$

When this is true, we say that closed-loop stability of the continuous-time feedback system (1.1) and (1.2) is *robust with respect to sampling*. This robustness holds in the finite-dimensional case (see, for example, [1]) and forms the basis for so-called continuous-time design, sampled-data implementation control methodologies.

We are interested in the infinite-dimensional case where the issue is rather more delicate. First, we need to make the formulation more precise.

We consider (1.1) as a state-space system with state-space X and input space U, where X and U are both Hilbert spaces. We assume that A is the generator of a strongly continuous semigroup $T(t)$ on X, and B is a linear bounded operator from U into X_{-1}. Here, X_{-1} is the completion of X in the norm $\|x\|_{-1} = \|(\lambda I - A)^{-1}x\|$ where λ is any element of the resolvent set of A. If the input operator B maps boundedly into the state-space X, then B is called bounded, otherwise B is called "unbounded" (with respect to the state-space X). See [8] for details about unbounded input operators and the extrapolation space X_{-1}.

In [4], it was shown that, in general, if the feedback operator is noncompact, then stability of (1.1) and (1.2) is not robust with respect to sampling. Here, we recall the simple counterexample in [4].

Example 1.1: Consider a system with $X = l^2(\mathbb{Z})$, $A = \mathrm{diag}_{k\in\mathbb{Z}}\,(1+ki)$, $B = I$, and $F = -2I$. Then, $A + BF$ generates the exponentially stable semigroup

$$T_{BF}(t) = \mathrm{diag}_{k\in\mathbb{Z}}\left(e^{(-1+ki)t}\right)$$

whereas the sampled-data feedback control given by (1.3) results in an evolution of $x(k\tau) =: x_k$ of the sampled-data feedback system (1.1) and (1.3) according to

$$x_{k+1} = \left(\mathrm{diag}_{k\in\mathbb{Z}}\left(e^{(1+ki)\tau}\right) - 2\mathrm{diag}_{k\in\mathbb{Z}}\left(\frac{e^{(1+ki)\tau}-1}{1+ki}\right)\right)x_k. \tag{1.5}$$

Now

$$\left|e^{(1+ki)\tau} - 2\left(\frac{e^{(1+ki)\tau}-1}{1+ki}\right)\right| \geq e^{\tau} - \frac{2}{|1+ik|}(e^{\tau}+1)$$

so we see that for any $\tau > 0$, the eigenvalues λ_k^τ of the sampled system (1.5) satisfy

$$\liminf_{|k|\to\infty} |\lambda_k^\tau| \geq e^{\tau} > 1.$$

Hence, the sampled-data feedback system is unstable for all $\tau > 0$.

For this reason, we restrict attention to systems with compact feedbacks F. Now sampled-data stabilization of infinite-dimensional systems with compact feedback operators is possible only if (A, B) satisfies certain necessary conditions. These necessary conditions are recalled here from [6, Th. 1.3], Theorem 1.3 for clarity:

Theorem 1.2: Suppose the sampled-data feedback system (1.1) and (1.3) is exponentially stable. Then, X admits a decomposition $X_s \oplus X_u$ which satisfies the following conditions.

a) $\dim X_u < \infty$.
b) $AX_s \subseteq X_s$ and $AX_u \subseteq X_u$.
c) There exists $\beta < 0$ so that (i) $\sigma(A|_{X_u}) = \sigma(A) \cap \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > \beta\}$ consists of at most finitely many eigenvalues of A, each with finite algebraic multiplicity and ii) $\sigma(A|_{X_s}) = \sigma(A) \cap \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < \beta\}$, and $A|_{X_s}$ is the generator of an exponentially stable semigroup.
d) Let $P: X \to X_u$ be the projection of X onto X_u along X_s, and

$$B_d^\tau = \int_0^\tau T(s)BF\,ds.$$

Then, the finite-dimensional discrete-time system $(T(\tau)|_{X_u}, PB_d^\tau)$ is controllable.

These necessary conditions for robustness of stabilization with respect to sampling when F is compact restrict the class of infinite-dimensional systems quite considerably. Within this class, it is proved in [4] that feedback stabilization is robust with respect to sampling in two important cases, namely when B is bounded or when B need not be bounded but A generates an analytic semigroup. (Note that when B is

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unbounded, the operator $A+BF$ has to be interpreted carefully.) More precisely, we have the following.

Theorem 1.3: Assume that A generates a strongly continuous semigroup $T(t)$ on X, $B \in \mathcal{B}(U, X)$, $F \in \mathcal{B}(X, U)$ is compact, and the semigroup generated by $A + BF$ is exponentially stable. Then, there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$, there exist $N \geq 1$ and $\nu > 0$ such that all solutions of (1.1) and (1.3) satisfy $\|x(t)\| \leq Ne^{-\nu t}\|x^0\|$ for all $x^0 \in X$ and all $t \geq 0$.

We note that the systems in Theorem 1.3 satisfy the necessary conditions in Theorem 1.2; see, for instance, [5].

Theorem 1.4: Assume that A generates an analytic semigroup $T(t)$ on X, $B \in \mathcal{B}(U, X_{-1})$, $F \in \mathcal{B}(X, U)$ is compact. If the semigroup generated by A_{BF} (as defined in [4, Section 4] and which agrees with $A + BF$ on the domain of A_{BF}) is exponentially stable, then there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$, there exist $N \geq 1$ and $\nu > 0$ such that all solutions of (1.1) and (1.3) satisfy $\|x(t)\| \leq Ne^{-\nu t}\|x^0\|$ for all $x^0 \in X$ and all $t \geq 0$.

We see from the counterexample for noncompact F given in Example 1.1, the necessary conditions from Theorem 1.2, and the sufficient conditions in Theorems 1.3 and 1.4, that robustness with respect to sampling is quite delicate. One obvious question is whether these sufficient conditions can be relaxed further. Now the gap between the necessary conditions and the sufficient conditions is quite subtle. There are systems which satisfy the necessary conditions but not the sufficient conditions, and we can draw no conclusions about the robustness of these systems with respect to sampling. We give an example which satisfies the necessary conditions of Theorem 1.2 and for which stability is not robust with respect to sampling.

II. Counterexample

In this section, we construct an example which satisfies the necessary conditions in Theorem 1.2 and shows that Theorem 1.3 is not true when B is not bounded, and Theorem 1.4 is not true when the semigroup generated by A is not analytic. In the example U is one-dimensional and B is "barely unbounded," in a sense made precise in the following.

To study the stability of the sampled-data system (1.1) and (1.3) we consider the power stability of a related discrete-time operator. Integrating (1.1) and (1.3) over one sampling interval $[k\tau, (k+1)\tau)$, and setting $x_k := x(k\tau)$ yields the discrete-time system

$$x_{k+1} = \Delta_\tau x_k, \text{ where } \Delta_\tau := T(\tau) + \int_0^\tau T(s)BF\,ds. \tag{2.1}$$

It follows from [6, Lemma 2.1] that Δ_τ is a bounded operator from X to X.

Proposition 2.1: ([4, Lemma 2.3]) For any $\tau > 0$, (1.1) and (1.3) is exponentially stable if and only if Δ_τ is power stable, i.e., $\|\Delta_\tau^k\| \to 0$ as $k \to \infty$.

Our counterexample is described as follows.

Proposition 2.2: Consider a system (1.1) and (1.2) defined as follows:

Let $X = \ell^2$, indexed by $\mathbb{N}$, with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and $U = \mathbb{C}$.

Define the generator $A = \operatorname{diag}(\lambda_k)_{k\in\mathbb{N}}$, with

$$\lambda_k := -1 + i\pi 3^k \text{ for } k \in \mathbb{N}$$

and domain $\mathcal{D}(A) = \{(x_k)_{k\in\mathbb{N}} \mid (x_k\lambda_k)_{k\in\mathbb{N}} \in \ell^2\}$. Let $B = (b_k)_{k\in\mathbb{N}}$ with

$$b_k := ik \text{ for } k \in \mathbb{N}.$$

Finally, define the operator $F \in \mathcal{B}(X, U)$ by

$$F(x_k) = \sum_{k\in\mathbb{N}} \frac{2x_k}{k}.$$

Since A is diagonal and invertible, we can easily define a branch of $A^{-\delta}$ for any $\delta > 0$.

Theorem 2.3: For the system given in Example 2.2 we have

1) (A, B) satisfies the necessary conditions a)–d) in Theorem 1.2;
2) B has one-dimensional range, is not bounded, but $A^{-\delta}B$ is bounded for all $\delta > 0$;
3) F is bounded and rank one;
4) $A + BF$ generates an exponentially stable semigroup;
5) there does not exist $\tau^* > 0$ so that Δ_τ given by (2.1) is power stable for all $\tau \in (0, \tau^*)$.

The condition that $B \in \mathcal{B}(U, X_{-1})$ becomes $A^{-1}B \in \mathcal{B}(U, X)$. One common measure of the unboundedness of B is the smallest $\delta \in \mathbb{R}$ such that $A^{-\delta}B \in \mathcal{B}(U, X)$. If we use this as a measure of unboundedness of B, then we see from condition 2) that Theorem 1.3 is sharp.

Now 1)–3) in Theorem 2.3 are clearly satisfied. Indeed, since A is exponentially stable, it is obvious that (A, B) satisfies the necessary conditions a)–d) in Theorem 1.2, so condition 1) is satisfied. Conditions 2) and 3) are readily verified.

It remains for us to verify conditions 4) and 5).

First, note that since $F \in \mathcal{B}(X, U)$ and $B \in \mathcal{B}(U, X_{-1})$, for every $s \in \rho(A)$ we can define $FR(s, A)B$ and it is easy to verify that

$$FR(s, A)B = \sum_{k\in\mathbb{N}} \frac{2i}{s - \lambda_k}.$$

To check 4) and 5) we start by working with a related system $(A, \tilde{B}, \tilde{F})$ where $\tilde{B} = (\tilde{b}_k)_{k\in\mathbb{N}}$ with

$$\tilde{b}_k := i \quad \text{for } k \in \mathbb{N}$$

and $\tilde{F}: X \supset \mathcal{D}(\tilde{F}) \to X$ is defined by

$$\tilde{F}(x_k) = \sum_{k\in\mathbb{N}} 2x_k \quad \mathcal{D}(\tilde{F}) = \ell^1.$$

It is easy to verify that for every $s \in \rho(A)$, $R(s, A)\tilde{B} \in \mathcal{D}(\tilde{F})$, and $\tilde{F}R(s, A)\tilde{B} = FR(s, A)B$.

For every $x \in \mathcal{D}(\tilde{F})$ we can compute $Ax + \tilde{B}\tilde{F}x$ in X_{-1}. The natural domain of $\mathcal{D}(A + \tilde{B}\tilde{F})$ is then

$$\mathcal{D}(A + \tilde{B}\tilde{F}) = \{x \in \mathcal{D}(\tilde{F}) \mid Ax + \tilde{B}\tilde{F}x \in X\}.$$

For $\alpha \in \mathbb{R}$, let

$$\mathbb{C}_\alpha := \{z \in \mathbb{C} \mid \operatorname{Re} z > \alpha\}.$$

Lemma 2.4: $A + BF$ has no eigenvalues in $\mathbb{C}_{-0.5}$.

Proof: We first show that for $x \in \mathcal{D}(A + \tilde{B}\tilde{F})$

$$\operatorname{Re}\langle (A + \tilde{B}\tilde{F})x, x\rangle = -\langle x, x\rangle. \tag{2.2}$$

Let $\tilde{A} := A + I$. We verify (2.2). If $x \in \mathcal{D}(\tilde{A} + \tilde{B}\tilde{F}) = \mathcal{D}(A + \tilde{B}\tilde{F})$, then $x \in \mathcal{D}(\tilde{F})$, so that $\sum_{m\in\mathbb{N}} x_m < \infty$, and

$$(\tilde{A} + \tilde{B}\tilde{F})x = \left(i\pi 3^k x_k + 2i \sum_{m\in\mathbb{N}} x_m\right)_{k\in\mathbb{N}} \in X = \ell^2.$$

Since $\ell^2 \subset \ell^\infty$ and $\left(2i\sum_{m\in\mathbb{N}} x_m\right)_{k\in\mathbb{N}} \in \ell^\infty$ it follows that $(3^k x_k)_{k\in\mathbb{N}} \in \ell^\infty$ and so $x_k = \mathcal{O}(3^{-k})$. Using this, we see that

$$\begin{aligned}\langle(\tilde{A}+\tilde{B}\tilde{F})x, x\rangle &= \sum_{k\in\mathbb{N}}\left(i\pi 3^k x_k + 2i\sum_{m\in\mathbb{N}} x_m\right) \\ \overline{x_k} &= i\left(\sum_{k\in\mathbb{N}} 3^k |x_k|^2 + 2\left|\sum_{k\in\mathbb{N}} x_k\right|^2\right)\end{aligned}$$

so

$$\mathrm{Re}\langle(\tilde{A}+\tilde{B}\tilde{F})x, x\rangle = 0.$$

Using the definition of $\tilde{A}$, we immediately see that this implies (2.2).

Now suppose that there exists $s_0 \in \mathbb{C}_{-0.5}$ such that

$$FR(s_0, A)B = \tilde{F}R(s_0, A)\tilde{B} = 1. \tag{2.3}$$

In particular $R(s_0, A)\tilde{B}$ is nonzero, so by a standard argument this would imply that s_0 is an eigenvalue of $A + \tilde{B}\tilde{F}$ with eigenvector $R(s_0, A)\tilde{B}$. However, (2.2) implies that $A + \tilde{B}\tilde{F}$ does not have any eigenvalues in $\mathbb{C}_{-0.5}$, so we see that there does not exist s_0 which solves (2.3).

Suppose that $A+BF$ has an eigenvalue $s \in \mathbb{C}_{-0.5}$ with eigenvector x. Then by a standard argument $FR(s, A)BFx = Fx$. If $Fx = 0$, then s is an eigenvalue of A which is in $\mathbb{C}_{-0.5}$, which we know is not the case, by the definition of A. Therefore, $FR(s, A)B = 1$, which contradicts the fact that (2.3) does not have a solution $s_0 \in \mathbb{C}_{-0.5}$. Hence, $A + BF$ does not have any eigenvalues in $\mathbb{C}_{-0.5}$. □

Remark 2.5: It is possible to prove the following two facts but they are more than we need: 1) $A + \tilde{B}\tilde{F}$ is maximally dissipative, hence, by the Lumer–Phillips Theorem it is the generator of an exponentially stable semigroup; and 2) $(A, \tilde{B}, \tilde{F})$ is a regular system, I is an *admissible feedback operator* for $(A, \tilde{B}, \tilde{F})$ (see [9]), and the closed-loop generator one obtains from this regular systems approach is $A + \tilde{B}\tilde{F}_L$, where $\tilde{F}_L$ is the *Lebesgue extension* of $\tilde{F}$, defined in [9]. The domain of $\tilde{F}_L$ might be larger than the domain of $\tilde{F}$ given in this note, but $A+\tilde{B}\tilde{F}$ and $A + \tilde{B}\tilde{F}_L$ have the same domain and are the same operator.

The following result completes the proof of condition (4) in Theorem 2.3.

Proposition 2.6: $A + BF$ generates an exponentially stable semigroup.

We first need to prove that $A + BF$ is a *discrete spectral operator*, and that all but finitely many of its eigenvalues have spectral projections with one-dimensional range; see [2] for the definition of a discrete spectral operator. To this end, we will apply [2, Th. XIX2.7 and Cor. XIX2.8] to $(A + BF)^* = A^* + F^*B^*$. Here, $\mathcal{D}(B^*) = \{(x_k)_{k\in\mathbb{N}} \mid \sum_{k\in\mathbb{N}} |kx_k|^2 < \infty\}$. The following hypotheses for this theorem are obviously true:

i) A^* is a discrete spectral operator with $\sigma(A^*) = \{\overline{\lambda_k}\}_{k\in\mathbb{N}}$;
ii) the spectral projection for each $\overline{\lambda_k} \in \sigma(A^*)$ has one-dimensional range;
iii) $\mathcal{D}(F^*B^*) \supseteq \mathcal{D}((A^*)^\nu) = \{x_k | \sum |x_k||\lambda_k|^\nu < \infty\}$ for every $\nu > 0$.

Let d_n be the distance from $\overline{\lambda_n}$ to $\sigma(A^*) \setminus \{\overline{\lambda_n}\}$, so $d_n = 2(3^{n-1})$. Then

$$\sum_{n\in\mathbb{N}} d_n^{-1}\left(|\lambda_n| + d_n\right)^\nu$$

converges for any $\nu < 1$. Hence, by [2, Th. XIX2.7], we conclude that $A^*+F^*B^*$ is a discrete spectral operator. Also, by [2, Cor. XIX2.8], all but finitely many $\mu_n \in \sigma(A^* + F^*B^*)$ have one dimensional spectral projection. The same properties hold for $A + BF$.

Since all but finitely many of the eigenvalues of $A + BF$ are simple and $A+BF$ is a discrete spectral operator, the nonsimple eigenvectors are of finite mulitiplicty, and this multiplicity is bounded. Now we can proceed using an argument found in [10]: $A + BF$ can be written in Jordan canonical form as follows. Let X_1 be the subspace of X spanned by the eigenvectors associated with the simple eigenvalues, and let X_2 be the finite-dimensional subspace of X spanned by the generalized eigenvectors associated with the remaining eigenvalues. Then $(A + BF)|_{X_1}$ can be written as a diagonal operator S_1, and $(A + BF)|_{X_2}$ can be written in the Jordan form $S_2 + N$, where S_2 is diagonal and N is nilpotent, with entries only on the superdiagonal. From this it follows that $e^{(S_1+S_2)t}$ has the same growth bound as $e^{(A+BF)t}$. From Lemma 2.4 we have that all of the eigenvalues of $A + BF$ are in $\{\mathrm{Re}\ s < -1/2\}$, so $e^{(S_1+S_2)t}$ and $e^{(A+BF)t}$ are both exponentially stable. □.

To complete the proof of Theorem 2.3 all that remains is to check that condition 5) holds for Example 2.2.

Theorem 2.7: There does not exist $\tau^* > 0$ so that Δ_τ is power stable for all $\tau \in (0, \tau^*)$.

Proof: We show that there exists (τ_n) such that $\lim_{n\to\infty} \tau_n = 0$, and Δ_{τ_n} has an eigenvalue z_n with $|z_n| > 1$.

Let $\mathbf{H}_\tau$ denote the discrete-time transfer function for the system

$$\begin{aligned}x_{k+1} &= e^{A\tau}x_k + \left(\int_0^\tau e^{A(\tau-\sigma)}B\, d\sigma\right)u_k \\ y_k &= Fx_k.\end{aligned}$$

It is easy to compute that

$$\mathbf{H}_\tau(z) = -\sum_{k\in\mathbb{N}} \frac{e^{\lambda_k\tau} - 1}{e^{\lambda_k\tau} - z}\frac{2i}{\lambda_k} \tag{2.4}$$

If z is a zero of $1 - \mathbf{H}_\tau(z)$, then z is an eigenvalue of Δ_τ (although the converse need not hold).

Let

$$\tau_n = 3^{-n} \text{ for } n \text{ even.}$$

Then, we write

$$-\mathbf{H}_{\tau_n}(z) = h_{1,n}(z) + h_{2,n}(z) + h_{3,n}(z) + h_{4,n}(z)$$

where

$$\begin{aligned}h_{1,n} &:= \sum_{k=1}^{n/2} \frac{e^{i\pi 3^{k-n}}e^{-3^{-n}} - 1}{e^{i\pi 3^{k-n}}e^{-3^{-n}} - z}\left(\frac{2i}{-1+i\pi 3^k}\right) \\ h_{2,n} &:= \sum_{k=(n/2)+1}^{n-1} \frac{e^{i\pi 3^{k-n}}e^{-3^{-n}} - 1}{e^{i\pi 3^{k-n}}e^{-3^{-n}} - z}\left(\frac{2i}{-1+i\pi 3^k}\right) \\ h_{3,n} &:= \sum_{k=n}^{\infty} \frac{e^{i\pi 3^{k-n}}e^{-3^{-n}} - 1}{e^{i\pi 3^{k-n}}e^{-3^{-n}} - z}\left(\frac{2i}{i\pi 3^k}\right) \\ h_{4,n} &:= \sum_{k=n}^{\infty} \frac{e^{i\pi 3^{k-n}}e^{-3^{-n}} - 1}{e^{i\pi 3^{k-n}}e^{-3^{-n}} - z}\left(\frac{2i}{i\pi 3^k(-1+i\pi 3^k)}\right).\end{aligned}$$

We will show that there exists a root of $1-\mathbf{H}_{\tau_n}(z)$ outside of the unit circle by showing that $h_{3,n}(z)+1$ has a root outside of the unit circle, and then applying Rouche's Theorem. For $k \geq n$, $\exp(i\pi 3^{k-n}) = -1$, so

$$\begin{aligned}h_{3,n}(z) &= \left(\frac{-e^{-3^{-n}} - 1}{-e^{-3^{-n}} - z}\right)\frac{2}{\pi}\sum_{k=n}^{\infty}\frac{1}{3^k} \\ &= \left(\frac{-e^{-3^{-n}} - 1}{-e^{-3^{-n}} - z}\right)\frac{3}{\pi 3^n}.\end{aligned} \tag{2.5}$$

Solving for z_n in $h_{3,n}(z_n) = -1$, we obtain

$$z_n = -\left(e^{-3^{-n}} + \frac{3}{\pi 3^n}\left(e^{-3^{-n}} + 1\right)\right) \tag{2.6}$$

so $z_n = -f(3^{-n})$, where

$$f(a) = e^{-a} + \left(\frac{3}{\pi}\right) a(e^{-a} + 1).$$

Now $f(0) = 1$ and $f(a) > 1$ for all $a > 0$. Therefore, for each $n \in \mathbb{N}$, $h_{3,n}(z) + 1$ has a zero z_n with $z_n < -1$.

We now show that $1 - \mathbf{H}_{\tau_n}(z)$ has a root outside of the unit circle for all sufficiently large n. Let

$$c_n := e^{-3^{-n}} - 1 + \frac{3}{\pi 3^n}(e^{-3^{-n}} + 1) \tag{2.7}$$

which is the distance from z_n to -1. Define the curve

$$\gamma_n := \left\{ z_n + \left(\frac{c_n}{2}\right) e^{i\theta} \mid 0 \le \theta < 2\pi \right\}.$$

If we can show that for sufficiently large n

$$|h_{3,n}(z) + 1| > |h_{1,n}(z) + h_{2,n}(z) + h_{4,n}(z)| \text{ for } z \in \gamma_n \tag{2.8}$$

then we can conclude from Rouche's Theorem that $1 - \mathbf{H}_{\tau_n}$ has a zero $\tilde{z}_n$ inside γ_n. Since all points on γ_n are outside of the unit disk, this will complete the proof of Theorem 2.7. We break up the proof of (2.8) into four steps.

Step 1) In this step, we prove that there exists $c > 0$, independent of n sufficiently large, such that

$$|h_{3,n}(z) + 1| > c \text{ for } z \in \gamma_n. \tag{2.9}$$

To prove this, we compute that for $z = z_n + (c_n/2)e^{i\theta}$, using (2.5)–(2.7)

$$|h_{3,n}(z) + 1| = \frac{1}{2}\left|\frac{c_n 3^n \pi}{3(e^{-3^{-n}} + 1) - \left(\frac{3^n c_n}{2}\right)\pi e^{i\theta}}\right|. \tag{2.10}$$

Since

$$\lim_{n\to\infty} c_n 3^n = \left(\frac{6}{\pi}\right) - 1 \tag{2.11}$$

we see that the right-hand side of (2.10) is larger than or equal to a function of n which converges to $(6-\pi)/(18-\pi)$, proving (2.9) for sufficiently large n.

Step 2) In this step, we prove that

$$\lim_{n\to\infty} |h_{1,n}(z)| = 0 \text{ for } z \in \gamma_n. \tag{2.12}$$

To prove (2.12), note that

$$\left|e^{i\pi 3^{k-n}} e^{-3^{-n}} - 1\right|^2 = 1 + e^{-2(3^{-n})} - 2e^{-3^n}\cos(\pi 3^{k-n}).$$

We see from this that the maximum of $|e^{i\pi 3^{k-n}} e^{-3^{-n}} - 1|$ for $k = 1, 2, \ldots n/2$ and n fixed occurs when $k = n/2$, so

$$\left|e^{i\pi 3^{k-n}} e^{-3^{-n}} - 1\right| \le \left|e^{i\pi 3^{-n/2}} e^{-3^{-n}} - 1\right| \text{ for } k = 1, 2, \ldots, \frac{n}{2}. \tag{2.13}$$

Also, note that for $z = z_n + (c_n/2)e^{i\theta}$

$$\begin{aligned}\left|e^{i\pi 3^{k-n}} e^{-3^{-n}} - z\right| &= \left|\left(e^{i\pi 3^{k-n}} + 1\right)e^{-3^{-n}} + \frac{3}{\pi 3^n}(e^{-3^{-n}} + 1) - \frac{c_n}{2}e^{i\theta}\right| \\ &\ge e^{-3^{-n}} - \frac{3}{\pi 3^n}(e^{-3^{-n}} + 1) - \frac{c_n}{2} \text{ for } \\ k &= 1, 2, \ldots, \frac{n}{2}.\end{aligned} \tag{2.14}$$

Using (2.13) and (2.14), we see that for large enough n

$$\begin{aligned}|h_{1,n}(z)| &\le \sum_{k=1}^{n/2}\left|\frac{e^{i\pi 3^{k-n}} e^{-3^{-n}} - 1}{e^{i\pi 3^{k-n}} e^{-3^{-n}} - z}\right|\left|\frac{2}{-1 + i\pi 3^k}\right| \\ &\le \frac{\left|e^{i\pi 3^{-n/2}} e^{-3^{-n}} - 1\right|}{e^{-3^{-n}} - \left(\frac{3}{\pi 3^n}\right)(e^{-3^{-n}} + 1) - \left(\frac{c_n}{2}\right)} \sum_{k=1}^{\infty}\frac{2}{\pi 3^k}.\end{aligned}$$

Since $\lim_{n\to\infty} c_n = 0$, is clear that (2.12) is true.

Step 3) In this step, we show that

$$\lim_{n\to\infty} |h_{2,n}(z)| = 0 \text{ for } z \in \gamma_n. \tag{2.15}$$

To prove (2.15), note that

$$\left|e^{i\pi 3^{k-n}} e^{-3^{-n}} - 1\right| \le 2 \text{ for } k = \frac{n}{2}, \frac{n}{2} + 1, \ldots, n - 1. \tag{2.16}$$

Also note that by the same argument that led to (2.14), we obtain for $z = z_n + (c_n/2)e^{i\theta}$

$$\begin{aligned}|e^{i\pi 3^{k-n}} e^{-3^{-n}} - z| &\ge e^{-3^{-n}} - \frac{3}{\pi 3^n}(e^{-3^{-n}} + 1) - \frac{c_n}{2} \text{ for } \\ k &= \frac{n}{2}, \frac{n}{2} + 1, \ldots, n - 1.\end{aligned} \tag{2.17}$$

Using (2.16) and (2.17)

$$\begin{aligned}|h_{2,n}(z)| &\le \sum_{k=n/2}^{n-1}\left|\frac{e^{i\pi 3^{k-n}} e^{-3^{-n}} - 1}{e^{i\pi 3^{k-n}} e^{-3^{-n}} - z}\right|\left|\frac{2}{-1 + i\pi 3^k}\right| \\ &\le \left|\frac{2}{e^{-3^{-n}} - \left(\frac{3}{\pi 3^n}\right)(e^{-3^{-n}} + 1) - \left(\frac{c_n}{2}\right)}\right| \sum_{k=n/2}^{\infty} \\ &\quad \times \frac{2}{\pi 3^k}.\end{aligned}$$

The last sum goes to zero as $n \to \infty$, so this proves (2.15).

Step 4) In this step, we show that

$$\lim_{n\to\infty} |h_{4,n}(z)| = 0 \text{ for } z \in \gamma_n. \tag{2.18}$$

To prove (2.18), note that for $z = z_n + (c_n/2)e^{i\theta}$

$$\begin{aligned}|h_{4,n}(z)| &\le \left|\frac{-e^{-3^{-n}} - 1}{-e^{-3^{-n}} - z_n - \left(\frac{c_n}{2}\right)e^{i\theta}}\right| \sum_{k=n}^{\infty} \\ &\quad \times \left|\frac{2}{(-1 + i\pi 3^k)(i\pi 3^k)}\right| \\ &\le \frac{2}{\left|(e^{-3^{-n}} + 1)\left(\frac{3}{\pi 3^n}\right) - \left(\frac{c_n}{2}\right)e^{i\theta}\right|}\frac{1}{\pi 3^n}\sum_{k=n}^{\infty} \\ &\quad \times \frac{2}{|-1 + i\pi 3^k|}.\end{aligned} \tag{2.19}$$

Now, using (2.11), we have that

$$\begin{aligned}&\frac{2}{\left|(e^{-3^{-n}} + 1)\left(\frac{3}{\pi 3^n}\right) - \left(\frac{c_n}{2}\right)e^{i\theta}\right|}\frac{1}{\pi 3^n} \\ &\qquad \le \frac{4}{(e^{-3^{-n}} + 1)6 - c_n \pi 3^n} \to \frac{4}{6 + \pi}\end{aligned}$$

as $n \to \infty$. Since the last sum in (2.19) goes to zero as $n \to \infty$, this proves (2.18).

Combining (2.9), (2.12), (2.15), and (2.18), we see that (2.8) is true for all sufficiently large n. As noted earlier, this completes the proofs of Theorems 2.7 and 2.3. □

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The Polynomial Approach to the LQ Non-Gaussian Regulator Problem

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***Abstract*—A new approach for the solution of the regulator problem for linear discrete-time dynamical systems with non-Gaussian disturbances is proposed. This approach generalizes a previous result concerning the definition of the quadratic optimal regulator. It consists in the definition of the polynomial optimal algorithm of order ν for the solution of the linear quadratic non-Gaussian stochastic regulator problem for systems with partial state information. The validity of the separation principle has also been proved in this case. Numerical simulations show the high performance of the proposed method with respect to the classical linear regulation techniques.**

***Index Terms*—Kalman filter, Linear quadratic Gaussian (LQG) optimal control, non-Gaussian systems, separation principle, stochastic control.**

I. INTRODUCTION

In this note, the optimal regulator problem for linear stochastic non-Gaussian discrete-time systems, with a quadratic cost criterion and partial state information, is considered. The approach proposed here consists in the polynomial extension of the solution previously given in [1], that concerned the definition of the quadratic optimal regulator. In a considerable number of technical areas, the widely used Gaussian assumption must be removed to obtain a more realistic statistical description of the disturbances acting on the state equations and/or the measurement process. As shown by various papers (e.g., [2]), increasing attention is being paid to non-Gaussian systems in control engineering, where both parameter and state estimation problems are important in designing feedback control laws. Non-Gaussian problems often arise in digital communications when the noise interference includes noise components that are essentially non-Gaussian (this is a common situation at frequencies below 100 MHz) [3]. Neglecting these components is a major source of error in communication systems design.

The deterministic version of this problem has received much attention in the scientific literature (see, for example, [4]–[8]). For stochastic systems, some authors assume that only partial information about the statistics of noises is available (minimax [8]–[10] or H_∞ [11] approaches). Other authors assume that the noise statistics are Gaussian (LQG control problem [12]–[16]). It is well known that, for Gaussian linear stochastic systems with incomplete state observation, this problem is equivalent to the simultaneous solution of a control and a filtering problem (separation principle [17]).

The purpose of this note is to propose a new algorithm for solving the optimal stochastic regulator problem for non-Gaussian systems with incomplete state observation (i.e., only noisy state observations are available) when a quadratic index is considered. The resulting solution is given by the same feedback control law as in the linear optimal regulator [18], [19], and by a filtering stage given by the optimal polynomial filter [20]. As a by product, it will be also proven an extension of the separation principle for non-Gaussian control systems when a suitable class of suboptimal state estimators (with respect to the conditional expectation) is considered.

In Section II, the precise statement of the problem is given. Then, it will be recalled the optimal solution of the problem in the Gaussian case [18], [19], that represents also a suboptimal linear solution for the non-Gaussian setting. The main result of the note is given in Section III, where the optimal polynomial regulator for a linear discrete-time non-Gaussian system is given. In Section IV, some numerical simulations are presented showing the better performance of the proposed polynomial optimal control with respect to the linear one. The note ends with concluding remarks in Section V.

II. PROBLEM FORMULATION AND PRELIMINARIES

The systems here considered are described by the following equations:

$$\begin{aligned} x(k+1) =& A(k)x(k) + B(k)u(k) + F(k)N(k) \\ &+ d(k) \quad x(0) = \bar{x} \end{aligned} \tag{2.1}$$

$$y(k) = C(k)x(k) + G(k)N(k) \tag{2.2}$$

with the associate quadratic index

$$J = \frac{1}{2}E\left\{x^T(N)S(N)x(N) + \sum_{k=0}^{N-1}(x^T(k)Q(k)x(k) + u^T(k)R(k)u(k))\right\} \tag{2.3}$$

where, for any k, $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^q$, $u(k) \in \mathbb{R}^p$, $N(k) \in \mathbb{R}^n$, $d(k) \in \mathbb{R}^n$, $F(k) \in \mathbb{R}^{n\times n}$, and $G(k) \in \mathbb{R}^{q\times n}$. Throughout this note, we will use the basic notations, symbols and rules of the Kronecker algebra (see [20]). Among these, let M and N be any matrices, the symbol $M \otimes N$ indicates the Kronecker product of M and N; the notation $M^{[l]} = M \otimes M^{[l-1]}$, $l \geq 1$, indicates the l-order Kronecker power of M, and $st(M)$ represents the stack operator applied to M. The noise $N(k)$ forms a sequence of non-Gaussian random vector variables, with all moments up to the 2νth order finite and known:

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