Internal model based tracking and disturbance rejection for stable well-posed systems

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Abstract: In this paper we solve the tracking and disturbance rejection problem for infinite-dimensional linear systems, with reference and disturbance signals that are finite superpositions of sinusoids. We explore two approaches, both based on the internal model principle. In the first approach, we use a low gain controller, and here our results are a partial extension of results by T. Hämäläinen and S. Pohjolainen. In their papers, the plant is required to have an exponentially stable transfer function in the Callier-Desoer algebra, while in this paper we only require the plant to be well-posed and exponentially stable. These conditions are sufficiently unrestrictive to be verifiable for many partial differential equations in more than one space variable. Our second approach concerns the case when the second component of the plant transfer function (from control input to tracking error) is positive. In this case, we identify a very simple stabilizing controller which is again an internal model, but which does not require low gain. We apply our results to two problems involving systems modeled by partial differential equations: the problem of rejecting external noise in a model for structure/acoustics interactions, and a similar problem for two coupled beams.

Keywords: well-posed linear system; tracking; internal model principle; input-output stability; exponential stability; dynamic stabilization; positive transfer function; optimizability; structural acoustics; coupled beams.

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1. Introduction and statement of the main results

In this paper we solve a tracking and disturbance rejection problem for stable well-posed linear systems, using a low gain controller suggested by the internal model principle. These results are a partial extension of those in Hämäläinen and Pohjolainen [10] (detailed comments on this connection will be given). Our results may also be regarded as a generalization of certain results from Logemann and Townley [19], who consider controllers with one pole on the imaginary axis (at zero), while we allow several such poles, corresponding to several relevant frequencies.

We assume that the plant Σ_p is a well-posed linear system and it is exponentially stable (see Section 2 for the concepts). The plant has two inputs, w and u. The input w consists of the external signals (references and disturbances) and u is the control input. These signals take values in the Hilbert spaces W and U, respectively. The output signal of Σ_p, denoted by z, which represents the tracking error, takes values in the Hilbert space Y. The transfer function of the plant is

$$\mathbf{P} = [\mathbf{P}_1 \;\; \mathbf{P}_2],$$

where $\mathbf{P}_1(s) \in \mathcal{L}(W, Y)$ and $\mathbf{P}_2(s) \in \mathcal{L}(U, Y)$. Let Σ_c be the well-posed controller which is to be determined. The closed-loop system obtained by interconnecting these systems is shown in Figure 1.

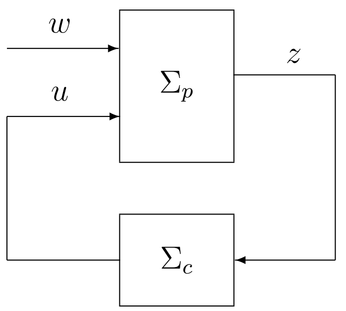

Figure 1: The closed-loop system built by interconnecting the stable well-posed plant Σ_p with the well-posed controller Σ_c. The signal w contains disturbances and references, and z is the error signal which should be made small. The transfer functions of Σ_p and Σ_c are $\mathbf{P} = [\mathbf{P}_1 \;\; \mathbf{P}_2]$ and $\mathbf{C}$.

We denote the state space of Σ_p by X (this is a Hilbert space). The state trajectories x of this system satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = Ax(t) + B_1 w(t) + B_2 u(t), \tag{1.1}$$

where A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X and B_1 and B_2 are admissible control operators for $\mathbb{T}$. This follows from the general representation theory of well-posed linear systems, which will be briefly recalled in Section 2. The exponential stability of Σ_p means that $\|\mathbb{T}_t\|$ decays to zero at an exponential rate. If Σ_p is regular (regular systems are a subclass of well-posed systems, see Section 2), then z (the output function of Σ_p) is given by

$$z(t) = C_\Lambda x(t) + D_1 w(t) + D_2 u(t), \tag{1.2}$$

for almost every $t \geq 0$. Here, C_Λ is an unbounded operator from X to Y (defined in Section 2), $D_1 \in \mathcal{L}(W,Y)$ and $D_2 \in \mathcal{L}(U,Y)$. In this case,

$$\mathbf{P}_1(s) = C_\Lambda(sI - A)^{-1}B_1 + D_1, \qquad \mathbf{P}_2(s) = C_\Lambda(sI - A)^{-1}B_2 + D_2\,.$$

If Σ_p is not regular, then (1.2) has to be replaced by a more complicated formula, see (2.8) in Section 2. A similar description applies to Σ_c.

For technical reasons, we consider two additional artificial input signals injected into the feedback system in Figure 1, u_{in} and z_{in}, which are added to u and z, as shown in Figure 2. These signals could be interpreted as additional disturbances, noise, measurement errors or quantization errors. We consider the output signals to be z_{out} and u_{out} (the output signals of Σ_p and Σ_c). Thus, from the three inputs to the two outputs, we have six relevant transfer functions. These, arranged in a 2×3 matrix, form the transfer function of the closed-loop system.

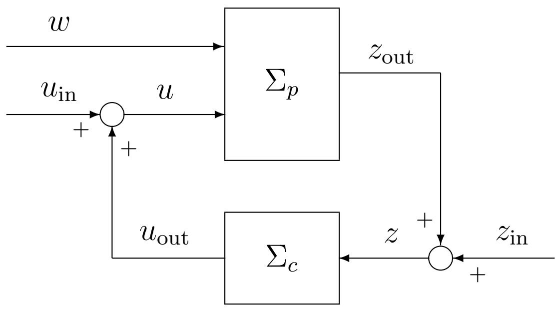

Figure 2: The closed-loop system from Figure 1, with two additional input signals u_{in} and z_{in}, which could be interpreted as noise or as measurement errors.

The controller Σ_c will be chosen such that the closed-loop system in Figure 2 (with inputs w, u_{in} and z_{in} and outputs z_{out} and u_{out}) is well-posed. In this case, the closed-loop system is described by equations similar to (1.1) and (1.2), and its state space is the Cartesian product of the state spaces of Σ_p and Σ_c (see [30, 37]).

Let $\mathcal{J}$ be a finite index set of integers. We assume that w is of the form

$$w(t) = \sum_{j \in \mathcal{J}} \mathrm{w}_j e^{i\omega_j t}, \quad \mathrm{w}_j \in W, \quad \omega_j \in \mathbb{R}. \tag{1.3}$$

Thus, w is a superposition of constant and sinusoidal signals. The frequencies ω_j are assumed to be known (for design purposes), but the vectors w_j (which determine the amplitudes and the phases) are not known in advance.

We introduce some terminology and notation. For any $a \in \mathbb{R}$, we put

$$\mathbb{C}_a := \{s \in \mathbb{C} \mid \operatorname{Re} s > a\}.$$

For any Banach space Z, we denote by $H_a^\infty(Z)$ the space of bounded analytic Z-valued functions on $\mathbb{C}_a$. For $a = 0$ we also use the notation $H^\infty(Z)$. In the sequel, when the space Z is clear from the context, we write H_a^∞ for $H_a^\infty(Z)$ and H^∞ for $H^\infty(Z)$. A transfer function is called *stable* if it is in H^∞, and *exponentially stable* if it is in H_a^∞ for some $a < 0$. If a well-posed system is exponentially stable, then its transfer function is also exponentially stable, see Section 2. In particular, for the system in Figure 1, there exists $a < 0$ such that

$$\mathbf{P}_1 \in H_a^\infty(\mathcal{L}(W,Y)), \qquad \mathbf{P}_2 \in H_a^\infty(\mathcal{L}(U,Y)).$$

For any $\alpha \in \mathbb{R}$ we denote

$$L^2_\alpha[0,\infty) = \left\{ f \in L^2_{loc}[0,\infty) \;\middle|\; \int_0^\infty e^{-2\alpha t}|f(t)|^2 \mathrm{d}t < \infty \right\}.$$

The corresponding space of Y-valued functions is denoted by $L^2_\alpha([0,\infty),Y)$ (or also $L^2([0,\infty),Y)$ if $\alpha = 0$). However, by some abuse of notation, we sometimes just write $L^2_\alpha[0,\infty)$ when the range space Y is clear from the context.

The objective of this paper is to find a controller Σ_c with transfer function $\mathbf{C}$ so that the closed-loop system in Figure 1 is exponentially stable, and the output z (the tracking error) decays exponentially to zero, by which we mean that $z \in L^2_\alpha[0,\infty)$ for some $\alpha < 0$. From an engineering point of view, it would look more convincing if we would require that $\lim_{t\to\infty} z(t) = 0$. However, such an objective is unrealistic due to the very general class of plants that we consider. Indeed, the output of Σ_p resulting from a non-smooth initial state may be in L^2_α without any further continuity properties, so that point-evaluations of z do not make sense. However, the condition $z \in L^2_\alpha[0,\infty)$ with $\alpha < 0$ actually expresses a very rapid decay of the output, since the sequence $E_n = \int_n^{n+1} \|z(t)\|^2 \mathrm{d}t$ decays exponentially.

We now recall the main result of Hämäläinen and Pohjolainen [10]. Their block diagram is slightly different, it corresponds to taking $\mathbf{P}_1 = [-I \;\; \mathbf{P}_2]$ in Figure 1 (see further comments on this after Figure 2), and they consider U and Y to be finite-dimensional, but these are not essential restrictions. They consider $\mathbf{P}$ to be an exponentially stable transfer function in the Callier-Desoer algebra. The internal model principle of Davison, Wonham and Francis (see [6, 7, 8]) suggests that $\mathbf{C}$ should have poles at $\{i\omega_j \mid j \in \mathcal{J}\}$. Following this principle, the following controller transfer function has been proposed and analyzed in [10]:

$$\mathbf{C}(s) = -\varepsilon \sum_{j\in\mathcal{J}} \frac{K_j}{s - i\omega_j}, \tag{1.4}$$

with

$$K_j \in \mathcal{L}(Y,U)\,, \qquad \sigma(\mathbf{P}_2(i\omega_j)K_j) \subset \mathbb{C}_0\,. \tag{1.5}$$

Note that (1.5) implies that $\mathbf{P}_2(i\omega_j)K_j$ is invertible, hence the range of $\mathbf{P}_2(i\omega_j)$ is Y (i.e., the matrix $\mathbf{P}_2(i\omega)$ is onto at the relevant frequencies). It was shown in [10] that for all $\varepsilon > 0$ sufficiently small, the feedback system in Figure 1 (with the assumptions of [10] just described) has exponentially stable transfer functions and moreover, if w is as in (1.3), then the error z tends to zero. The approach of [10] is algebraic and they consider also multiple poles in $\mathbf{C}$, which are needed if we want to allow the coefficients w_j in (1.3) to be polynomials in t (we have not reproduced the formulas corresponding to the multiple poles, since we only consider constant w_j).

This result is important because it allows tracking and disturbance rejection for external signals that are superpositions of a constant and several sinusoidal signals, *with very little information about the plant.* Indeed, all we have to know is that the plant is stable and we must have some knowledge about $\mathbf{P}_2(i\omega_j)$, which can be measured, but we do not need to be very precise (since the conditions (1.5) are robust with respect to small changes of $\mathbf{P}_2(i\omega_j)$). By contrast, the internal model theory developed in many references requires a detailed knowledge of the plant equations. We feel that this is an important and beautiful result, and we wanted to understand it from a different perspective, using an analytic approach.

How is our setup different from the one in [10]: the small differences in the block diagram and in the spaces U, Y have already been mentioned (in both respects, our framework is slightly more general). We have constant coefficients w_j in (1.3), so we consider only simple poles for $\mathbf{C}$, and in this respect we are less general. Most importantly, we replace the class of plants in [10] with exponentially stable well-posed plants, which is substantially more general than the class of stable plants with transfer functions in the Callier-Desoer algebra. The conditions for a plant to be well-posed are sufficiently unrestrictive to be verifiable for many partial differential equations in more than one space variable; this is probably not true if we have to verify that the transfer is in the Callier-Desoer algebra.

Our first main results follows. The concepts of exact controllability and exact observablity, which appear in the theorem, will be recalled in Section 2.

Theorem 1.1. *Suppose that Σ_p is an exponentially stable well-posed linear system with transfer function $\mathbf{P} = [\mathbf{P}_1\ \mathbf{P}_2]$, where $\mathbf{P}_1(s) \in \mathcal{L}(W,Y)$, $\mathbf{P}_2(s) \in \mathcal{L}(U,Y)$. Let Σ_c be an exactly controllable and exactly observable realization of a transfer function $\mathbf{C}$ of the form*

$$\mathbf{C}(s) = \; -\,\varepsilon \left(\mathbf{C}_0(s) + \sum_{j\in\mathcal{J}} \frac{K_j}{s - i\omega_j} \right), \tag{1.6}$$

where $\mathbf{C}_0 \in H^\infty_\alpha(\mathcal{L}(Y,U))$ with $\alpha < 0$, $K_j \in \mathcal{L}(Y,U)$ and $\sigma(\mathbf{P}_2(i\omega_j)K_j) \subset \mathbb{C}_0$.

Then the closed-loop system shown in Figure 2 is well-posed and exponentially stable for all sufficiently small $\varepsilon > 0$. For any such ε there exists $\beta < 0$ such that, if w is of the form (1.3), then $z \in L^2_\beta[0,\infty)$.

It follows from Theorem 1.1 that (for sufficiently small $\varepsilon > 0$) the six transfer functions of the closed-loop system in Figure 2 are in H^∞_β for some $\beta < 0$.

Note that, in (1.6), we have added the extra term $\mathbf{C}_0$ (when compared to (1.4)). The theorem would of course remain valid without this extra term, but $\mathbf{C}_0$ may be needed to satisfy some other design requirements, possibly derived from robustness considerations, or to shorten the transient response.

This theorem has not been stated in the strongest possible form: it is a consequence of a more general theorem stated and proved in Section 3. In the more general version, the conditions "exactly controllable" and "exactly observable" are replaced with the less restrictive conditions "optimizable" and "estimatable". The reason for not stating this version here is that the concepts of optimizability and estimatability would probably be less familiar to most readers. If U and Y are finite-dimensional and $\mathbf{C}$ is rational (as it would be in most applications), then it is natural to take Σ_c to be a minimal realization of $\mathbf{C}$, so that Σ_c would be controllable and observable, as required in the theorem.

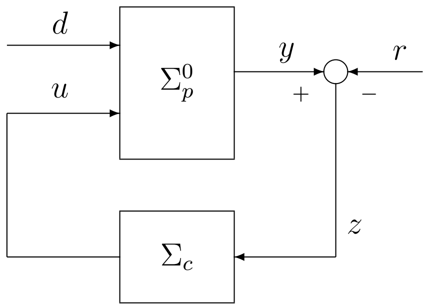

Figure 3: The stable well-posed plant Σ_p^0 interconnected with the well-posed controller Σ_c. The transfer functions of these systems are $\mathbf{P}^0 = [\mathbf{P}_1^0 \;\; \mathbf{P}_2]$ and $\mathbf{C}$. If we lump together the disturbance d and the reference r to form the signal w, then this block diagram reduces to the one in Figure 1, with $\mathbf{P}_1 = [-I \;\; \mathbf{P}_1^0]$.

We make some comments on the generality of the block diagram in Figure 1 and the signal w in (1.3). Typically w will contain both the reference r and the disturbance d, i.e., $w = [r \;\; d]^\top$. In this case $r(t) \in Y$ and $d(t)$ is in another Hilbert space V, we have $W = Y \times V$ and $\mathbf{P}_1$ can be decomposed as $\mathbf{P}_1 = [-I \;\; \mathbf{P}_1^0]$, so that, neglecting the influence of the initial states of Σ_p and Σ_c,

$$\hat{z} = \mathbf{P}_1 \hat{w} + \mathbf{P}_2 \hat{u} = \; -\hat{r} + \mathbf{P}_1^0 \hat{d} + \mathbf{P}_2 \hat{u} = \hat{y} - \hat{r}.$$

Here, $\hat{y} = \mathbf{P}_1^0 \hat{d} + \mathbf{P}_2 \hat{u}$ is the Laplace transform of the output signal y of the plant Σ_p^0 with transfer function $\mathbf{P}^0 = [\mathbf{P}_1^0 \;\; \mathbf{P}_2]$, corresponding to the initial state zero. The system Σ_p^0 is almost the same as Σ_p (it has the same state trajectories) but its inputs are only d and u (instead of r, d and u), as shown in Figure 3. The paper [10] uses a block diagram which corresponds to this decomposition, with the additional restriction $\mathbf{P}_1^0 = \mathbf{P}_2$. In the theory that we develop, such a decomposition of w and of $\mathbf{P}_1$ is not needed. We have assumed that the plant is exponentially stable, but in applications Σ_p might be obtained by stabilizing an initially unstable plant. This is illustrated in our Section 5 (disturbance rejection for a coupled beams system).

Usually, W is the complexification of a real Hilbert space W_0, so that any $\mathrm{w} \in W$ has a unique decomposition $\mathrm{w} = \mathrm{w}^0 + i\mathrm{w}^1$ with $\mathrm{w}^0, \mathrm{w}^1 \in W_0$ and so the complex conjugate $\overline{\mathrm{w}} = \mathrm{w}^0 - i\mathrm{w}^1$ is well defined. Moreover, the signal w takes values in W_0, which (with proper indexing) implies that

$$\omega_{-j} = -\omega_j, \quad \text{and} \quad \mathrm{w}_{-j} = \overline{\mathrm{w}}_j .$$

Similarly, U and Y are usually the complexifications of real Hilbert spaces U_0 and Y_0 and the transfer function $\mathbf{P}$ is *real*, which means that

$$\mathbf{P}_1(-i\omega)\mathrm{w}_0 = \overline{\mathbf{P}_1(i\omega)\mathrm{w}_0} \qquad \forall\, \mathrm{w}_0 \in W_0 ,$$

and a similar condition is satisfied by $\mathbf{P}_2$. In this case, the controller transfer function $\mathbf{C}$ can be chosen to be real as well, by choosing

$$K_{-j}\mathrm{y}_0 = \overline{K_j \mathrm{y}_0} \qquad \forall\, \mathrm{y}_0 \in Y_0 ,$$

and by choosing $\mathbf{C}_0$ to be real. These restrictions on w, $\mathbf{P}$ and $\mathbf{C}$ had to be mentioned because they would probably arise in any engineering application. However, these restrictions are not necessary for our theory, so that we will not make them. Taking them into account would not modify the theory at all.

The results of Hämäläinen and Pohjolainen [10] build on earlier work by the same authors, notably [22] and [9]. Low-gain tracking results for infinite-dimensional systems can be found in Logemann, Bontsema, and Owens [14], Logemann and Curtain [15], Logemann and Mawby [16], Logemann and Owens [17], Logemann and Townley [19, 20], Logemann, Ryan and Townley [18], Pohjolainen [22, 23], and Pohjolainen and Lätti [24]. In [15, 16, 18, 19, 20] the reference and disturbance signals are constant and the class of plants considered is the class of regular systems, which is a large subset of well-posed systems. In [15, 16, 18] the systems are allowed to have nonlinearities. In [19] the plants are uncertain and it is shown how the gain can be chosen adaptively. In [20] analogous results are obtained for uncertain discrete-time systems, with the gain obtained adaptively; these results are applied to continuous-time regular systems to obtain adaptive sampled-data low-gain tracking controllers. The references [14, 17, 22, 23, 24] are earlier works where the classes of systems considered are substantially more restrictive than here, and the references and disturbances are constant.

Let $\mathbf{P}$ be an $\mathcal{L}(U)$-valued transfer function defined on (a set containing) the half-plane $\mathbb{C}_0$. We say that $\mathbf{P}$ is a *positive* transfer function if

$$\operatorname{Re} \mathbf{P}(s) := \frac{1}{2}\left[\mathbf{P}(s) + \mathbf{P}(s)^*\right] \geq 0 \qquad \forall\, s \in \mathbb{C}_0 .$$

When the second component of the plant transfer function (from control input to error) is positive, the following theorem states that certain simple controllers will stabilize the system in the sense of Theorem 1.1 and also achieve tracking. Moreover, in this situation, there is no need to adjust an unknown small gain.

Theorem 1.2. *Suppose that Σ_p is an exponentially stable well-posed linear system with transfer function $\mathbf{P} = [\mathbf{P}_1\ \mathbf{P}_2]$, where $\mathbf{P}_1(s) \in \mathcal{L}(W, U)$, $\mathbf{P}_2(s) \in \mathcal{L}(U)$, $\mathbf{P}_2$ is a positive transfer function, and $\operatorname{Re}\mathbf{P}_2(i\omega_j)$ is invertible for all $j \in \mathcal{J}$. Let Σ_c be an exactly controllable and exactly observable realization of a transfer function $\mathbf{C}$ of the form*

$$\mathbf{C}(s) = -\left(\mathbf{C}_0(s) + \sum_{j\in\mathcal{J}} \frac{K_j}{s - i\omega_j}\right), \tag{1.7}$$

where $K_j \in \mathcal{L}(U)$, $K_j > 0$, $K_j^{-1} \in \mathcal{L}(U)$, $\mathbf{C}_0 \in H^\infty_\alpha(\mathcal{L}(U))$ with $\alpha < 0$ and

$$\operatorname{Re}\mathbf{C}_0(s) \geq \frac{1}{2}I \qquad \forall\, s \in \mathbb{C}_0 .$$

Then the feedback system in Figure 2 is well-posed and exponentially stable. If w is of the form (1.3), then $z \in L^2_\beta[0,\infty)$ for some $\beta < 0$.

Concerning the conditions "exactly controllable" and "exactly observable" that appear in this theorem, the same comments apply as those made after Theorem 1.1. The more general version of Theorem 1.2 is stated and proved in Section 3.

We remark that a simple choice for K_j and $\mathbf{C}_0$ is to take them scalar multiples of the identity operator. Then $\mathbf{C}$ is essentially a single input-single output system repeated in identical copies as many times as the dimension of U. This choice still allows $\mathbf{C}_0$ to be a function of s, but of course the simplest choice for $\mathbf{C}_0$ would be to take it a real constant larger than 0.5.

In Section 4 we apply Theorem 1.1 to a model for structure/acoustics interaction, showing that external noise at fixed frequencies can be rejected. In Section 5 we apply Theorem 1.2 to a model for coupled Euler-Bernoulli beams.

2. Well-posed linear systems and related concepts

In this section we recall some basic facts about admissible control and observation operators, well-posed linear systems, their transfer functions, static output feedback, exponential stability and its relation to optimizability and estimatability.

Throughout this section, X is a Hilbert space and $A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{T}$ on X. The Hilbert space X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$ is fixed (this norm is equivalent to the graph norm). The Hilbert space X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$. This space is isomorphic to $\mathcal{D}(A^*)^*$, and we have

$$X_1 \subset X \subset X_{-1}\,,$$

densely and with continuous embeddings. $\mathbb{T}$ extends to a semigroup on X_{-1}, denoted by the same symbol. The generator of this extended semigroup is an extension of A, whose domain is X, so that $A : X \to X_{-1}$.

Let U be a Hilbert space and $B \in \mathcal{L}(U, X_{-1})$. Then B is called an *admissible control operator* for $\mathbb{T}$ if it has the following property: If x is the solution of

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \tag{2.1}$$

with $x(0) = x_0 \in X$ and $u \in L^2([0,\infty), U)$, then $x(t) \in X$ for all $t \geq 0$ (see [33] for details). In this case, x is a continuous X-valued function of t. We have $x(t) = \mathbb{T}_t x_0 + \Phi_t u$, where $\Phi_t \in \mathcal{L}(L^2([0,\infty), U), X)$ is defined by

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} Bu(\sigma)\, d\sigma\,. \tag{2.2}$$

The above integration is done in X_{-1}, but the result is in X. The Laplace transform of x is

$$\hat{x}(s) = (sI - A)^{-1}\left[x_0 + B\hat{u}(s)\right].$$

B is called *bounded* if $B \in \mathcal{L}(U, X)$ (and unbounded otherwise).

Let Y be a Hilbert space and $C \in \mathcal{L}(X_1, Y)$. Then C is called an *admissible observation operator* for $\mathbb{T}$ if it has the following property: For some (hence, for every) $T > 0$ there exists a $k_T \geq 0$ such that

$$\int_0^T \|C\mathbb{T}_t x_0\|^2 \mathrm{d}t \leq k_T^2 \|x_0\|^2 \qquad \forall\, x_0 \in \mathcal{D}(A)\,, \tag{2.3}$$

see [34]. C is called *bounded* if it can be extended such that $C \in \mathcal{L}(X, Y)$. If $C \in \mathcal{L}(X_1, Y)$, then $C^* \in \mathcal{L}(Y, X_{-1}^d)$, where X_{-1}^d is the completion of X with respect to the norm $\|z\|_{-1}^d = \|(\beta I - A^*)^{-1}z\|$, with $\beta \in \rho(A^*)$ (we have identified Y with its dual). C is an admissible observation operator for $\mathbb{T}$ if and only if C^* is an admissible control operator for the dual semigroup $\mathbb{T}^*$.

We regard $L^2_{loc}([0,\infty), Y)$ as a Fréchet space with the seminorms being the L^2 norms on the intervals $[0, n]$, $n \in \mathbb{N}$. Then the admissibility of C means that there is a continuous operator $\Psi : X \to L^2_{loc}([0,\infty), Y)$ such that

$$(\Psi x_0)(t) = C\mathbb{T}_t x_0\,, \qquad \forall\, x_0 \in \mathcal{D}(A)\,. \tag{2.4}$$

The operator Ψ is completely determined by (2.4), because $\mathcal{D}(A)$ is dense in X. We define the Λ-extension of C by

$$C_\Lambda z = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} z\,, \tag{2.5}$$

where λ is real. The domain $\mathcal{D}(C_\Lambda)$ consists of all $z \in X$ for which the above limit exists. If we replace C by C_Λ, then formula (2.4) becomes true for all $x_0 \in X$ and for almost every $t \geq 0$. If $y = \Psi x_0$, then its Laplace transform is

$$\hat{y}(s) = C(sI - A)^{-1} x_0 . \tag{2.6}$$

By a *well-posed linear system* we mean a linear time-invariant system such that on any finite time interval, the operator from the initial state and the input function to the final state and the output function is bounded. The input, state and output spaces are Hilbert spaces, and the input and output functions are of class L^2_{loc}. For the detailed definition, background and examples we refer to Salamon [27], [28], Staffans [29], [32] or Weiss [35], [36].

We recall some facts about well-posed linear systems. Let Σ be such a system, with input space U, state space X and output space Y. We consider positive time, $t \geq 0$. The state trajectories of Σ satisfy the equation (2.1) and the comments from the beginning of this section apply. $\mathbb{T}$ is called the *semigroup* of Σ and B is called the *control operator* of Σ. If u is the input function of Σ, x_0 is its initial state and y is the corresponding output function, then

$$y = \Psi x_0 + \mathbb{F} u . \tag{2.7}$$

Here, Ψ is given by (2.4) and continuous extension to X, and C appearing in (2.4) is called the *observation operator* of Σ. Clearly, both B and C are admissible.

The operator $\mathbb{F} : L^2_{loc}([0,\infty), U) \to L^2_{loc}([0,\infty), Y)$ is easiest to represent using Laplace transforms. We do not distinguish between two analytic functions defined on different right half-planes if one is a restriction of the other. For some $a \in \mathbb{R}$, there exists a unique $\mathcal{L}(U,Y)$-valued function $\mathbf{G}$ in H^∞_a, called the *transfer function* of Σ, which determines $\mathbb{F}$ as follows: if $u \in L^2([0,\infty), U)$ and $y = \mathbb{F}u$, then y has a Laplace transform $\hat{y}$ and, for Re s sufficiently large,

$$\hat{y}(s) = \mathbf{G}(s) \hat{u}(s) .$$

This determines $\mathbb{F}$, since L^2 is dense in L^2_{loc}. $\mathbf{G}$ is in H^∞_a for any $a \in \mathbb{R}$ which is larger than the growth bound of the semigroup $\mathbb{T}$, see [35, Section 4]. In particular, if $\mathbb{T}$ is exponentially stable, then $\mathbf{G}$ is in H^∞_a for some $a < 0$ (recall that such a transfer function is called exponentially stable). An operator-valued analytic function defined on a domain containing a right half-plane and which is bounded on this half-plane is called a *well-posed transfer function.* We have

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C \left[(sI - A)^{-1} - (\beta I - A)^{-1} \right] B ,$$

for any s, β in the open right half-plane determined by the growth bound of $\mathbb{T}$. This shows that $\mathbf{G}$ is determined by A, B and C up to an additive constant operator. In addition to (2.7), there is also a formula which expresses y locally in time: for any complex β with $\operatorname{Re}\beta$ sufficiently large,

$$y(t) = C_\Lambda \left[x(t) - (\beta I - A)^{-1} B u(t) \right] + \mathbf{G}(\beta) u(t), \tag{2.8}$$

valid for almost every $t \geq 0$, where x is the state trajectory of Σ, see [32].

Any operator-valued well-posed transfer function $\mathbf{G}$ has a realization, i.e., a well-posed linear system with transfer function $\mathbf{G}$, see Salamon [28] and Staffans [31]. Moreover, it follows from the material in [28, 31] that if $\mathbf{G}$ is exponentially stable, then the realization can be chosen to be exponentially stable.

An operator $K \in \mathcal{L}(Y,U)$ is called an *admissible feedback operator* for Σ (or for $\mathbf{G}$) if $I - \mathbf{G}K$ has a well-posed inverse (equivalently, if $I - K\mathbf{G}$ has a well-posed inverse). If this is the case, then the system with output feedback shown in Figure 4 is well-posed (its input is v, its state and output are the same as for Σ). This new system is called the *closed-loop system* corresponding to Σ and K, and it is denoted by Σ^K. Its transfer function is $\mathbf{G}^K = \mathbf{G}(I-K\mathbf{G})^{-1} = (I-\mathbf{G}K)^{-1}\mathbf{G}$. We have that $-K$ is an admissible feedback operator for Σ^K and the corresponding closed-loop system is Σ. For more details on closed-loop systems we refer to [36].

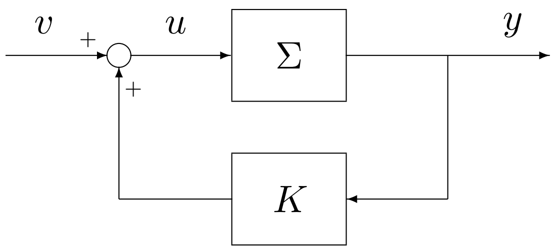

Figure 4: A well-posed linear system Σ with output feedback via K. If K is admissible, then this is a new well-posed linear system Σ^K, called the closed-loop system.

The system Σ is called *regular* if the limit

$$\lim_{s\to+\infty} \mathbf{G}(s)\mathrm{v} = D\mathrm{v}$$

exists for every $\mathrm{v} \in U$, where s is real (see [35]). In this case, the operator $D \in \mathcal{L}(U,Y)$ is called the *feedthrough operator* of Σ. Regularity is equivalent to the fact that the product $C_\Lambda(sI-A)^{-1}B$ makes sense. In this case,

$$\mathbf{G}(s) = C_\Lambda(sI-A)^{-1}B + D\,, \tag{2.9}$$

like in finite dimensions. This, together with (2.8), implies that the function y from (2.7) satisfies, for almost every $t \geq 0$,

$$y(t) = C_\Lambda x(t) + Du(t)\,, \tag{2.10}$$

where x is the state trajectory of the system. The operators A, B, C, D are called the *generating operators* of Σ, because they determine Σ via (2.1) and (2.10). (A,B,C) is called a *regular triple* if $A, B, C, 0$ are the generating operators of a regular linear system. Equivalently, A generates a semigroup, B and C are admissible, the product $C_\Lambda(sI-A)^{-1}B$ exists and it is bounded on some right half-plane (see Section 2 of

[37]). In particular, if A is a generator, one of B and C is admissible and the other is bounded, then (A, B, C) is a regular triple.

If, in the feedback system shown in Figure 4, Σ is regular and its feedthrough operator is zero, then the semigroup generator of Σ^K is

$$A^K = A + BKC_\Lambda, \qquad \mathcal{D}(A^K) = \{x \in \mathcal{D}(C_\Lambda) \mid Ax + BKC_\Lambda x \in X\}. \qquad (2.11)$$

In this case, Σ^K is regular and its other generating operators are

$$B^K = B, \qquad C^K = C_\Lambda|_{\mathcal{D}(A^K)}, \qquad D^K = 0, \qquad (2.12)$$

where $C_\Lambda|_{\mathcal{D}(A^K)}$ denotes the restriction of C_Λ to $\mathcal{D}(A^K)$. It is interesting to note that in this case $C^K_\Lambda = C_\Lambda$, see [36, Section 7] for details.

Definition 2.1. The well-posed system Σ (or the pair (A, B)) is *optimizable* if for every $x_0 \in X$ there exists a $u \in L^2([0, \infty), U)$ such that $x \in L^2([0, \infty), X)$, where x is the state trajectory defined by $x(t) = \mathbb{T}_t x_0 + \Phi_t u$. The system Σ (or the pair (A, B)) is *exactly controllable* if for some $t > 0$, Φ_t is onto.

Optimizability is the most natural extension of the the concept of stabilizability from finite-dimensional systems to the context of well-posed systems. Motivated by linear quadratic optimal control theory, this property is sometimes called *the finite cost condition*. For details concerning optimizability we refer to Weiss and Rebarber [38]. It is easy to see that exact controllability implies optimizability.

Definition 2.2. The well-posed system Σ (or the pair (A, C)) is *estimatable* if (A^*, C^*) is optimizable. The system Σ (or the pair (A, C)) is *exactly observable* if (A^*, C^*) is exactly controllable.

Estimatability is equivalent to the solvability of a certain final state estimation problem, see [38, Section 3]. Since it is the dual concept of optimizability, estimatability is an extension of the concept of detectability from finite-dimensional systems to the context of well-posed systems. Exact observability is equivalent to the fact that there exist $T > 0$ and $K_T > 0$ such that

$$\int_0^T \|C\mathbb{T}_t x_0\|^2 \mathrm{d}t \geq K_T \|x_0\|^2 \qquad \forall\, x_0 \in \mathcal{D}(A).$$

Exact observability implies estimatability. We refer again to [38] for further details.

It is easy to see that if a well-posed system is exponentially stable, then it is optimizable and estimatable. Optimizability and estimatability are invariant under static output feedback. One of the main results of [38] is the following:

Theorem 2.3. *A well-posed linear system is exponentially stable if and only if it is optimizable, estimatable and input-output stable.*

The following theorem is a consequence of the last theorem (related results for regular linear systems were given in Weiss and Curtain [37]).

Theorem 2.4. *Suppose that Σ_p and Σ_c are well-posed linear systems with transfer functions denoted $\mathbf{P}$ and $\mathbf{C}$ (where $\mathbf{P} = [\mathbf{P}_1\ \mathbf{P}_2]$), connected in feedback as shown in Figure 2. Suppose that Σ_p is optimizable via u, Σ_c is optimizable, and both systems are estimatable. Suppose that the four transfer functions*

$$(I - \mathbf{P}_2\mathbf{C})^{-1},\ \mathbf{C}(I - \mathbf{P}_2\mathbf{C})^{-1},\ (I - \mathbf{P}_2\mathbf{C})^{-1}\mathbf{P}_2,\ \mathbf{C}(I - \mathbf{P}_2\mathbf{C})^{-1}\mathbf{P}_2 \tag{2.13}$$

are all stable (i.e., in H^∞). Then the closed-loop system $\Sigma_{p,c}$ shown in Figure 2 is an exponentially stable well-posed linear system.

Proof. Under the given optimizability and estimatability hypotheses, it follows from [38, Theorem 6.4] that the closed-loop system $\Sigma_{p,c}$ is well-posed and exponentially stable if (and only if) $(I - \mathbf{L})^{-1} \in H^\infty(U \times Y)$, where

$$\mathbf{L} = \begin{bmatrix} 0 & \mathbf{C} \\ \mathbf{P}_2 & 0 \end{bmatrix}.$$

It is easy to verify that $(I - \mathbf{L})^{-1} \in H^\infty$ if (and only if) the transfer functions listed in (2.13) are all in H^∞. ■

The transfer functions appearing in (2.13) are closely related to the transfer functions of the system in Figure 2, with $w = 0$. For example, $(I - \mathbf{P}_2\mathbf{C})^{-1}$ is the transfer function from z_{in} to z, hence $(I - \mathbf{P}_2\mathbf{C})^{-1} - I$ maps z_{in} to z_{out}.

Remark 2.5. With the assumptions and the notation of Theorem 2.4, let γ denote the growth bound of the semigroup of $\Sigma_{p,c}$, so that $\gamma < 0$ according to the theorem. Then according to the theory recalled earlier, all of the transfer functions associated with this system (such as the four transfer functions listed in (2.13)) are in H^∞_β for any $\beta > \gamma$. Similarly, $\mathbf{G} = (I - \mathbf{P}_2\mathbf{C})^{-1}\mathbf{P}_1$, which is the transfer function from w to z for $\Sigma_{p,c}$, is in H^∞_β for any $\beta > \gamma$.

3. Proof of the main results

To prove the stability parts of Theorems 1.1 and 1.2 we show that an appropriate set of closed-loop transfer functions is stable, and then we apply Theorem 2.4 to conclude that the closed-loop system is exponentially stable.

To be able to apply Theorem 2.4, a key step is to show that the first of the four transfer functions listed in (2.13), called the *sensitivity* and denoted by $\mathbf{S}$, is stable (i.e., in H^∞) for all sufficiently small $\varepsilon > 0$.

Lemma 3.1. *Suppose that* $\mathbf{P}_2 \in H^\infty_a(\mathcal{L}(U,Y))$ *with* $a<0$ *and* $\mathbf{C}$ *is given by* (1.6) *and satisfies the conditions listed after* (1.6)*. Then there exists* $\varepsilon^*>0$ *such that for all* $\varepsilon \in (0,\varepsilon^*]$,

$$\mathbf{S} := (I-\mathbf{P}_2\mathbf{C})^{-1} \in H^\infty .$$

Proof: Let $r_0>0$ be such that $|\omega_j-\omega_k|>2r_0$ for all $j,k\in\mathcal{J}$ with $j\neq k$. We define the half-disks

$$D_j := \{s\in\mathbb{C} \mid \operatorname{Re} s\geq 0, \quad |s-i\omega_j|\leq r_0\} .$$

Since $\mathbf{P}_2$ and $\mathbf{C}_0$ are bounded on $\mathbb{C}_0$ and $\mathcal{J}$ is finite set, it is clear that

$$\sup_{s\in\mathcal{G}} \left\| \mathbf{P}_2(s)\left[\mathbf{C}_0(s)+\sum_{j\in\mathcal{J}}\frac{K_j}{s-i\omega_j}\right]\right\| < \infty ,$$

where

$$\mathcal{G} = \mathbb{C}_0 \setminus \cup_{j\in\mathcal{J}} D_j .$$

Therefore, there exists $\varepsilon_\infty>0$ such that

$$\mathbf{S}(s) = \left(I+\varepsilon\mathbf{P}_2(s)\left[\mathbf{C}_0(s)+\sum_{j\in\mathcal{J}}\frac{K_j}{s-i\omega_j}\right]\right)^{-1}$$

is uniformly bounded for all $s\in\mathcal{G}$ and all $\varepsilon\in(0,\varepsilon_\infty]$.

Fix $j\in\mathcal{J}$. To analyze $\mathbf{S}$ on D_j, we define

$$\mathbf{S}_j(s) := \left(I+\frac{\varepsilon\mathbf{P}_2(i\omega_j)K_j}{s-i\omega_j}\right)^{-1} .$$

(this is a good approximation of $\mathbf{S}(s)$ near the point $s=i\omega_j$) and

$$\mathbf{Q}_j(s) = \frac{(\mathbf{P}_2(s)-\mathbf{P}_2(i\omega_j))\,K_j}{s-i\omega_j} + \mathbf{P}_2(s)\mathbf{C}_0(s) + \sum_{k\in\mathcal{J},\,k\neq j}\frac{\mathbf{P}_2(s)K_k}{s-i\omega_k} .$$

It is clear that the last two terms in the definition of $\mathbf{Q}_j$ are bounded on D_j. Since $\mathbf{P}_2$ is analytic at $i\omega_j$, the first term in the definition of $\mathbf{Q}_j$ is also bounded on D_j. Therefore $\mathbf{Q}_j$ is bounded on D_j, with a bound that is independent of ε (because $\mathbf{Q}_j$ is independent of ε).

We have $\mathbf{S}^{-1}-\mathbf{S}_j^{-1}=\varepsilon\mathbf{Q}_j$, so that we can write

$$\mathbf{S}(s) = \mathbf{S}_j(s)\,(I+\varepsilon\mathbf{Q}_j(s)\mathbf{S}_j(s))^{-1} . \tag{3.1}$$

If we can show that $\mathbf{S}_j$ is bounded on D_j, with the bound independent of $\varepsilon>0$, then from (3.1) we see that there exists $\varepsilon_j>0$ such that $\mathbf{S}$ is bounded on D_j for all $\varepsilon\in(0,\varepsilon_j]$. Since $\mathcal{J}$ has finitely many elements, this will finish the proof, with

$$\varepsilon^* = \min\ \{\varepsilon_j \mid j\in\mathcal{J}\}\cup\{\varepsilon_\infty\} .$$

To show that $\mathbf{S}_j$ is bounded on D_j, with the bound independent of $\varepsilon > 0$, denote $P_j = \mathbf{P}_2(i\omega_j)K_j$, $z = (s - i\omega_j)/\varepsilon$, and note that

$$\sup_{s\in D_j} \|\mathbf{S}_j(s)\| = \sup_{\operatorname{Re} z\geq 0,\, |z|\leq r/\varepsilon} \left\| \left(I + \frac{P_j}{z} \right)^{-1} \right\|$$

$$\leq \sup_{\operatorname{Re} z\geq 0} \|z(zI + P_j)^{-1}\| .$$

Since $\sigma(P_j) \subset \mathbb{C}_0$, $z(zI + P_j)^{-1}$ is analytic on $\overline{\mathbb{C}_0}$. Moreover,

$$\lim_{|z|\to\infty} z(zI + P_j)^{-1} = I ,$$

so $z(zI + P_j)^{-1}$ is continuous on the one-point compactification of $\overline{\mathbb{C}_0}$, hence it is bounded there. Therefore $\mathbf{S}_j$ is bounded on D_j with a bound independent of ε, which (as explained earlier) finishes the proof of the lemma. ■

Theorem 1.1 is an immediate consequence of the following, more general theorem.

Theorem 3.2. *Let Σ_p be an exponentially stable well-posed linear system with transfer function $\mathbf{P} = [\mathbf{P}_1\ \mathbf{P}_2]$, where $\mathbf{P}_1(s) \in \mathcal{L}(W, Y)$, $\mathbf{P}_2(s) \in \mathcal{L}(U, Y)$. Let Σ_c be an optimizable and estimatable realization of a transfer function $\mathbf{C}$ of the form* (1.6)*, where $\mathbf{C}_0 \in H^\infty_\alpha(\mathcal{L}(Y, U))$ with $\alpha < 0$, $K_j \in \mathcal{L}(Y, U)$ and $\sigma(\mathbf{P}_2(i\omega_j)K_j) \subset \mathbb{C}_0$.*

Then there exists an $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$, the feedback system in Figure 2 is well-posed and exponentially stable. For such ε, there exists a $\beta < 0$ such that if w is of the form (1.3)*, then $z \in L^2_\beta[0, \infty)$.*

The simplest way to obtain an optimizable and estimatable realization of $\mathbf{C}$ is to take an exponentially stable realization of $-\varepsilon\mathbf{C}_0$ and connect it in parallel with a minimal (i.e., exactly observable and exactly controllable) realization of the rational transfer function $\mathbf{C} + \varepsilon\mathbf{C}_0$. Note that this minimal realization is not finite-dimensional in general, because the coefficients K_j are operators.

Proof. Since Σ_p is exponentially stable, we have $\mathbf{P} \in H^\infty_a$ with $a < 0$. According to Lemma 3.1, for $\varepsilon \in (0, \varepsilon^*]$ we have $(I - \mathbf{P}_2\mathbf{C})^{-1} \in H^\infty$. We need to show that for such ε, the other transfer functions listed in (2.13) are also in H^∞. Due to the stability of $\mathbf{P}_2$, this follows once we have shown that

$$\mathbf{C}(I - \mathbf{P}_2\mathbf{C})^{-1} \in H^\infty.$$

To prove this, we notice that by (1.6), $\lim_{s\to i\omega_j} (s - i\omega_j)\mathbf{C}(s) = -\varepsilon K_j$ for any $j \in \mathcal{J}$. This implies that

$$\lim_{s\to i\omega_j} \frac{1}{s - i\omega_j} \left[I - \mathbf{P}_2(s)\mathbf{C}(s)\right]^{-1}$$

$$= \left[\lim_{s\to i\omega_j} \mathbf{P}_2(s)(s - i\omega_j)\mathbf{C}(s) \right]^{-1} = -\frac{1}{\varepsilon} \left[\mathbf{P}_2(i\omega_j)K_j\right]^{-1} . \tag{3.2}$$

Note that $\mathbf{P}_2(i\omega_j)K_j$ is invertible according to (1.5). Now, using again the decomposition (1.6), we conclude that $\mathbf{C}(I-\mathbf{P}_2\mathbf{C})^{-1}$ has a finite limit at $i\omega_j$, for all $j \in \mathcal{J}$, so that $\mathbf{C}(I-\mathbf{P}_2\mathbf{C})^{-1}$ is bounded on a neighborhood $\mathcal{N}$ of the set $\{i\omega_j \mid j \in \mathcal{J}\}$. Since $(I-\mathbf{P}_2\mathbf{C})^{-1} \in H^\infty$ and $\mathbf{C}$ is uniformly bounded on $\mathbb{C}_0 \setminus \mathcal{N}$, it follows that $\mathbf{C}(I-\mathbf{P}_2\mathbf{C})^{-1} \in H^\infty$. As mentioned earlier, this implies that all of the transfer functions listed in (2.13) are in H^∞.

Because Σ_p is exponentially stable, it is optimizable and estimatable. Σ_c is optimizable and estimatable by hypothesis. Now we apply Theorem 2.4 to conclude that $\Sigma_{p,c}$ is a well-posed system and it is exponentially stable, so that the growth bound γ of its semigroup satisfies $\gamma < 0$.

Now we choose $\beta \in (\gamma, 0)$ and we prove that $z \in L^2_\beta[0,\infty)$. First we recall from Remark 2.5 that the transfer function $\mathbf{G}$ from w to z is in H^∞_β. From (3.2),

$$\lim_{s\to i\omega_j} \frac{1}{s-i\omega_j}\mathbf{G}(s) = -\frac{1}{\varepsilon}\left[\mathbf{P}_2(i\omega_j)K_j\right]^{-1}\mathbf{P}_1(i\omega_j). \tag{3.3}$$

According to (1.3), we have

$$\hat{w}(s) = \sum_{j\in\mathcal{J}} \frac{\mathrm{w}_j}{s-i\omega_j}.$$

This, together with (3.3), shows that $\mathbf{G}\hat{w}$ has removable singularities at the points $i\omega_j$, $j \in \mathcal{J}$. Since $\mathbf{G}$ is bounded on $\mathbb{C}_\beta$ and $\hat{w}$ decays like $1/s$ for large $|s|$, we see that $\mathbf{G}\hat{w} \in H^2_\beta(Y)$. Here $H^2_\beta(Y)$ denotes the space of those Y-valued analytic functions on $\mathbb{C}_\beta$ which can be obtained by shifting a function in the Hardy space $H^2(Y)$ (on $\mathbb{C}_0$) to the left by $|\beta|$. Equivalently, by the Paley-Wiener theorem, $H^2_\beta(Y)$ is the space of Laplace transforms of functions in $L^2_\beta([0,\infty),Y)$.

The output function z of $\Sigma_{p,c}$ can be decomposed as $z = z_1 + z_2$, where z_1 is the component due to the initial state and z_2 is the component due to the input w. According to [35, formula (4.2)], $z_1 \in L^2_\beta[0,\infty)$. Since $\hat{z}_2 = \mathbf{G}\hat{w} \in H^2_\beta$, we have that $z_2 \in L^2_\beta[0,\infty)$. Thus we have shown that $z \in L^2_\beta[0,\infty)$. ■

We now turn our attention to the proof of Theorem 1.2. We will need the following simple lemma, which is related to the computations usually found in the theory of the Cayley transform, but we are not aware of a reference for this lemma. Recall from Section 1 that for any operator $\mathbf{T} \in \mathcal{L}(H)$, we denote $\operatorname{Re}\mathbf{T} = \frac{1}{2}(\mathbf{T}+\mathbf{T}^*)$.

Lemma 3.3. *Suppose that H is a Hilbert space and $\mathbf{T} \in \mathcal{L}(H)$. Then*

$$\operatorname{Re}\mathbf{T} \geq \frac{1}{2}I$$

if and only if there exists $\mathbf{Q} \in \mathcal{L}(H)$ such that

$$\mathbf{T} = (I-\mathbf{Q})^{-1}, \quad \text{with} \quad \|\mathbf{Q}\| \leq 1. \tag{3.4}$$

Furthermore, if the above conditions are satisfied and we denote $M = \|\mathbf{T}\|$, then

$$\operatorname{Re}\mathbf{Q} \leq \left(1 - \frac{1}{2M^2}\right)I.$$

Proof. Suppose that $\operatorname{Re}\mathbf{T} \geq \frac{1}{2}I$. Then for any $\mathrm{u} \in H$ with $\|\mathrm{u}\| = 1$,

$$\|\mathbf{T}\mathrm{u}\| \geq |\langle \mathbf{T}\mathrm{u}, \mathrm{u}\rangle| \geq \operatorname{Re}\langle \mathbf{T}\mathrm{u}, \mathrm{u}\rangle = \langle (\operatorname{Re}\mathbf{T})\mathrm{u}, \mathrm{u}\rangle \geq \frac{1}{2},$$

so that $\mathbf{T}$ is bounded from below. By a similar argument, $\mathbf{T}^*$ is also bounded from below, hence $\mathbf{T}$ is onto, so that $\mathbf{T}$ is invertible. We have for any $\mathrm{v} \in H$

$$\begin{aligned}\|(\mathbf{T}-I)\mathrm{v}\|^2 &= \langle (\mathbf{T}-I)\mathrm{v}, (\mathbf{T}-I)\mathrm{v}\rangle \\ &= \|\mathbf{T}\mathrm{v}\|^2 - 2\operatorname{Re}\langle \mathbf{T}\mathrm{v}, \mathrm{v}\rangle + \|\mathrm{v}\|^2 \\ &\leq \|\mathbf{T}\mathrm{v}\|^2 - \|\mathrm{v}\|^2 + \|\mathrm{v}\|^2 = \|\mathbf{T}\mathrm{v}\|^2.\end{aligned}$$

Denoting $z = \mathbf{T}\mathrm{v}$, we have shown that

$$\|(\mathbf{T}-I)\mathbf{T}^{-1}z\| \leq \|z\| \qquad \forall\, z \in H, \tag{3.5}$$

so that $\mathbf{Q} = (\mathbf{T}-I)\mathbf{T}^{-1} = I - \mathbf{T}^{-1}$ is a contraction. It is clear that $\mathbf{T} = (I-\mathbf{Q})^{-1}$

Conversely, suppose that (3.4) holds. Then obviously $\mathbf{T}$ is invertible and $\mathbf{Q} = (\mathbf{T}-I)\mathbf{T}^{-1}$, so that (3.5) holds. Denoting $\mathrm{v} = \mathbf{T}^{-1}z$, we ge

$$\|(\mathbf{T}-I)\mathrm{v}\| \leq \|\mathbf{T}\mathrm{v}\| \qquad \forall\, \mathrm{v} \in H.$$

Developing $\|(\mathbf{T}-I)\mathrm{v}\|^2$ as in the first part of the proof, we obtain that $2\operatorname{Re}\langle \mathbf{T}\mathrm{v}, \mathrm{v}\rangle \geq \|\mathrm{v}\|^2$, which is the same as $\operatorname{Re}\mathbf{T} \geq \frac{1}{2}I$.

The final part of the lemma follows from

$$\mathbf{Q}^* + \mathbf{Q} = 2I - \mathbf{T}^{-*} - \mathbf{T}^{-1} = 2I - \mathbf{T}^{-*}(\mathbf{T}^* + \mathbf{T})\mathbf{T}^{-1} \leq 2I - \mathbf{T}^{-*}\mathbf{T}^{-1}.$$

If $\|\mathbf{T}\| = M$ then $\mathbf{T}^{-*}\mathbf{T}^{-1} \geq \frac{I}{M^2}$, so that

$$\mathbf{Q}^* + \mathbf{Q} \leq 2I - \frac{I}{M^2}. \qquad \blacksquare$$

Theorem 1.2 is an immediate consequence of the following, more general theorem.

Theorem 3.4. *Suppose that Σ_p is an exponentially stable well-posed linear system with transfer function $\mathbf{P} = [\mathbf{P}_1\ \mathbf{P}_2]$, where $\mathbf{P}_1(s) \in \mathcal{L}(W,U)$, $\mathbf{P}_2(s) \in \mathcal{L}(U)$, $\mathbf{P}_2$ is a positive transfer function, and $\operatorname{Re}\mathbf{P}_2(i\omega_j)$ is invertible for all $j \in \mathcal{J}$. Let Σ_c be an optimizable and estimatable realization of a transfer function $\mathbf{C}$ of the form (1.7), where $K_j \in \mathcal{L}(U)$, $K_j > 0$, $K_j^{-1} \in \mathcal{L}(U)$, $\mathbf{C}_0 \in H^\infty_\alpha(\mathcal{L}(U))$ with $\alpha < 0$ and*

$$\operatorname{Re}\mathbf{C}_0(s) \geq \frac{1}{2}I \qquad \forall\, s \in \mathbb{C}_0.$$

Then the feedback system in Figure 2 is well-posed and exponentially stable. If w is if the form (1.3), then $z \in L^2_\beta[0,\infty)$ for some $\beta < 0$.

Proof. For every $K_j \in \mathcal{L}(U)$ with $K_j \geq 0$ and every $\omega_j \in \mathbb{R}$,

$$\mathrm{Re}\left(\frac{K_j}{s - i\omega_j}\right) \geq 0 \qquad \forall\, s \in \mathbb{C}_0 .$$

Therefore, from the formula (1.7) we see that

$$-\mathrm{Re}\,\mathbf{C}(s) \geq \frac{1}{2} I\,, \tag{3.6}$$

which implies that for all $s \in \mathbb{C}_0$

$$\|\mathbf{C}(s)\mathrm{v}\| \geq \frac{1}{2}\|\mathrm{v}\| \qquad \forall\, \mathrm{v} \in U\,. \tag{3.7}$$

By Lemma 3.3, (3.6) implies that there exists $\mathbf{Q} \in \mathcal{L}(H)$ such that

$$-\mathbf{C} = (I - \mathbf{Q})^{-1}, \quad \|\mathbf{Q}\| \leq 1\,.$$

Since $\mathbf{S} = (I - \mathbf{P}_2\mathbf{C})^{-1}$, we obtain that

$$\mathbf{S}^{-1} = I + \mathbf{P}_2(I - \mathbf{Q})^{-1} = [I - \mathbf{Q} + \mathbf{P}_2](I - \mathbf{Q})^{-1} = -[I - \mathbf{Q} + \mathbf{P}_2]\mathbf{C}\,. \tag{3.8}$$

Since by our assumptions $\mathrm{Re}\,\mathbf{P}_2(i\omega_j)$ is positive and invertible for each $j \in \mathcal{J}$, there exists $\varepsilon > 0$ and $\delta_j > 0$ such that for all $j \in \mathcal{J}$,

$$\mathrm{Re}\,\mathbf{P}_2(s) > \varepsilon I \quad \text{for every} \quad s \in D_j := \{s \in \mathbb{C}_0 \mid |s - i\omega_j| < \delta_j\}\,.$$

Since $\mathrm{Re}\,\mathbf{Q} \leq I$ and $\mathrm{Re}\,\mathbf{P}_2 \geq \varepsilon I$ for $s \in D_j$, we see that $\mathrm{Re}\,[I - \mathbf{Q}(s) + \mathbf{P}_2(s)] \geq \varepsilon I$, so that

$$\|[I - \mathbf{Q}(s) + \mathbf{P}_2(s)]\mathrm{u}\| \geq \varepsilon\|\mathrm{u}\| \qquad \forall\, \mathrm{u} \in U\,.$$

Using this, together with (3.7) and (3.8), we obtain

$$\|\mathbf{S}^{-1}(s)\mathrm{v}\| \geq \frac{\varepsilon}{2}\|\mathrm{v}\| \qquad \forall\, \mathrm{v} \in U\,,\ s \in \cup_{j\in\mathcal{J}} D_j\,. \tag{3.9}$$

Furthermore, $\mathbf{C}$ is bounded on $\mathbb{C}_0 \setminus \cup_{j\in\mathcal{J}} D_j$, so from Lemma 3.3 there exists $\varepsilon_1 > 0$ such that $\mathrm{Re}\,\mathbf{Q} \leq (1 - \varepsilon_1)I$ in $\mathbb{C}_0 \setminus \cup_{j\in\mathcal{J}} D_j$. Since by hypothesis $\mathrm{Re}\,\mathbf{P}_2(s) \geq 0$ for $s \in \mathbb{C}_0$, $\mathrm{Re}\,(I - \mathbf{Q}(s) + \mathbf{P}_2(s)) \geq \varepsilon_1 I$ for $s \in \mathbb{C}_0 \setminus \cup_{j\in\mathcal{J}} D_j$. Therefore, using again the formulas (3.7) and (3.8), we obtain that

$$\|\mathbf{S}^{-1}(s)\mathrm{v}\| \geq \frac{\varepsilon_1}{2}\|\mathrm{v}\| \qquad \forall\, \mathrm{v} \in U\,,\ s \in \mathbb{C}_0 \setminus \cup_{j\in\mathcal{J}} D_j\,. \tag{3.10}$$

Combining (3.9) and (3.10), we see that $\mathbf{S}^{-1}$ is uniformly bounded from below on $\mathbb{C}_0$, i.e., $\mathbf{S}(s)$ is uniformly bounded on $\mathbb{C}_0$. Now we can follow the rest of the proof of Theorem 3.2, taking $\varepsilon = 1$. ∎

4. Noise reduction in a stuctural acoustics model

We consider a model for the following situation: A rectangle Ω represents a two-dimensional cross section of an acoustic cavity. An acoustic velocity potential z satisfies the wave equation in Ω. Γ_0 is one side of the boundary of Ω, and represents a cross section of the active wall of the cavity. We denote the displacement of this wall by v, which satisfies a beam equation with Kelvin-Voigt damping. The boundary conditions for the wave equation include boundary damping which causes the system energy to decay exponentially. The control action is realized by the placement of a piezoelectric ceramic patch on Γ_0; a voltage is applied on this patch and the resulting bending moments can be interpreted as derivatives of delta functions. Furthermore, there is a noise source $d(t)$ applied to the active wall, which we assume to be the sum of finitely many sinusoids; in an application the acoustic cavity might be an airplane cockpit and the noise might be engine noise. We refer to z_t as the *acoustic pressure*, although it is actually proportional to the acoustic pressure. We are interested in reducing the acoustic pressure near a point $\zeta_0 \in \Omega$, so we take the observation $y(t)$ to be an integral of the acoustic pressure over a small neighborhood $\Omega_0 \subset \Omega$ of ζ_0. We wish to design a controller to maintain the exponential stability of the system, and reject the effect of the noise $d(t)$ from the output. We will apply Theorem 1.1 to obtain this controller. This model has been studied in detail in Lasiecka [13, Chapters 4-7]. A tracking problem for a related structural acoustics model has been numerically studied in Banks *et al* [2].

Let $z = z(\zeta, t)$ for $t \in [0, \infty)$ and $\zeta \in \Omega$, let $v = v(\xi, t)$ for $t \in [0, \infty)$ and $\xi \in \Gamma_0$, and let $\partial/\partial\nu$ denote the outward normal derivative on Γ. Let $U = Y = \mathbb{R}$. In our motivating example b_2 will represent the action of one piezoelectric ceramic patch, and hence it will be of the form $b_2 = \alpha\delta'_{\eta_1} - \alpha\delta'_{\eta_2}$, where $\alpha \in \mathbb{R}$, $\eta_1, \eta_2 \in \Gamma_0$, and δ'_η is the (distributional) derivative of the Dirac delta function concentrated at η. We refer to Banks and Smith [3, eq. (2.3)] for a derivation of this input operator.

Let the disturbance d be of the form considered in Section 1,

$$d(t) = \sum_{j\in\mathcal{J}} \mathrm{d}_j e^{i\omega_j t}, \qquad \mathrm{d}_j \in \mathbb{R}, \qquad \omega_j \in \mathbb{R}, \tag{4.1}$$

where $\mathcal{J}$ is a finite index set. Let $b_1(\xi)$ be a function on Γ_0 which describes the spatial distribution on which the disturbance term $d(t)$ acts; for instance, if the noise is uniformly distributed throughout Γ_0, then $b_1(\xi) = 1$. We will assume for simplicity that $b_1 \in L^2(\Gamma_0)$, although the same procedure will work also for some distributions b_1 not in $L^2(\Gamma_0)$. We consider the following model:

$$\begin{aligned}
z_{tt} &= \Delta z \ \text{ on } [0,\infty)\times\Omega, \\
\frac{\partial z}{\partial\nu} + \beta z_t &= v_t \quad \text{on } [0,\infty)\times\Gamma_0, \\
\frac{\partial z}{\partial\nu} + \beta z_t &= 0 \quad \text{on } \ [0,\infty)\times\Gamma\setminus\Gamma_0, \qquad (4.2)\\
v_{tt} &= -D^4 v - D^4 v_t - z_t + b_1 d + b_2 u \ \text{ on } \ [0,\infty)\times\Gamma_0, \\
v(a,t) &= v(b,t) = 0, \quad (Dv)(a,t) = (Dv)(b,t) = 0,
\end{aligned}$$

where D is the derivative w.r.t. ξ operator, and $t \in [0, \infty)$. The corresponding output signal is

$$y(t) = \int_{\Omega_0} z_t(\zeta, t)\, d\zeta\,. \tag{4.3}$$

The damping, parameterized by $\beta > 0$, is either a natural part of the system, such as an energy-absorbing coating, or it would be introduced as a stabilizing feedback before the low-gain tracking is applied. This system fits into the framework described by Figure 3 (which is a special case of the system in Figure 1), where the reference signal r is taken to be zero, $\mathbf{P}_2$ is the transfer function from u to y, and $\mathbf{P}_1^0$ is the transfer function from d to y. For $d = 0$, this system is a special case of the one described by equations (5.2.1) and (5.3.23) in [13], and the results in [13] will allow us to verify the hypotheses of Theorem 1.1.

The state of this system is $x(t) = [z(t)\ z_t(t)\ v(t)\ v_t(t)]^\top$, and the state space with the appropriate compatability conditions is

$$X := \Big\{ [z_1\ z_2\ v_1\ v_2]^\top \in H^1(\Omega) \times L^2(\Omega) \times H_0^2(\Gamma_0) \times L^2(\Gamma_0) \Big| \\ \int_\Omega z_2\, d\zeta = \int_{\Gamma_0} [v_1 - z_1]\, d\sigma - \int_{\Gamma \setminus \Gamma_0} z_1\, d\sigma \Big\}.$$

From [13, Theorem 5.4.1], the equations (4.2) with $d = 0$ and $u = 0$ determine an exponentially stable semigroup $\mathbb{T}$ on X with generator A. (For an explicit description of A we refer to [13].)

The output signal y in (4.3) is obtained via a bounded observation functional C defined on X. If we denote $B_1 = [0\ 0\ 0\ b_1]^\top$ and $B_2 = [0\ 0\ 0\ b_2]^\top$, then (4.2) and (4.3) can be rewritten in the form

$$\dot{x}(t) = Ax(t) + B_1 d(t) + B_2 u(t), \qquad y(t) = Cx(t), \tag{4.4}$$

and B_1 is bounded (i.e., $B_1 \in X$). Hence, these equations determine a well-posed and regular system Σ_p if and only if the control operator B_2 is admissible for $\mathbb{T}$ (as explained in Section 2, after (2.10)). The admissibility of B_2 is established in [13, Prop. 6.6.1] when the damping parameter $\beta = 0$ and, as pointed out after [13, Remark 6.6.1], the addition of damping simplifies the arguments for proving [13, Prop. 6.6.1]. Thus, Σ_p is a well-posed, exponentially stable system, so we can apply Theorem 1.1 to it (we now have d and y in place of w and z).

We comment on the condition (1.5), which in our case reduces to $\mathbf{P}_2(i\omega_j)K_j \in \mathbb{C}_0$. Clearly it will be possible to choose the gains $\{K_j\}_{j\in\mathcal{J}}$ so that this is satisfied if and only if $\mathbf{P}_2(i\omega_j) \neq 0$ at all of the frequencies ω_j. Since $\mathbf{P}_2$ is analytic and not identically equal to zero in $\mathbb{C}_\alpha$ for $\alpha < 0$, we immediately conclude that the zeros of $\mathbf{P}_2$ on the imaginary axis have no finite accumulation point. Unfortunately, it is very difficult to compute $\mathbf{P}_2(i\omega_j)$ directly from (4.2) and (4.3); for this example the zeros of $\mathbf{P}_2$ would have to be estimated numerically. We will now prove that $\mathbf{P}_2(0) = 0$ (this implies that for this model constant disturbances cannot be rejected). If the initial state is zero, then (4.3) implies that

$$\hat{y}(s) = s \int_{\Omega_0} \hat{z}(\zeta, s)\, \mathrm{d}\zeta\,. \tag{4.5}$$

Let us denote the growth bound of $\mathbb{T}$ by γ, so that $\gamma < 0$. Choose $\beta \in (\gamma, 0)$, so that $\mathbf{P}_2 \in H^\infty_\beta$. Assume that $d = 0$, $u \in L^2_\beta[0, \infty)$ and $x(0) = 0$. Then the state trajectory x, given by $x(t) = \int_0^t \mathbb{T}_{t-\sigma} B_2 u(\sigma)\, d\sigma$, satisfies $\hat{x} \in H^2_\beta(X)$ (the definition of H^2_β was given after (3.3)). Since z is the first component of x, we have $\hat{z} \in H^2_\beta(H^1(\Omega))$. Now (4.5) shows that $\hat{y}$ is analytic in a neighborhood of 0 and $\hat{y}(0) = 0$. Since $\hat{y} = \mathbf{P}_2 \hat{u}$, we conclude that $\mathbf{P}_2(0) = 0$.

5. Disturbance rejection in a coupled beam

In this section we consider two coupled Euler-Bernoulli beams, not necessarily of equal length, with the disturbance acting through a term distributed along the beams. The disturbance has a finite number of sinusoidal components with known frequencies. There are two control inputs, acting at the joint. Our goal is to design a controller which exponentially stabilizes the system and rejects the disturbance. We mention the related work of Morgul [21], who considered a disturbance rejection problem for a single undamped beam with force input at one end. He showed that his closed-loop system is stable (strongly or exponentially, depending on a technical condition), but the decay rate of the error has not been considered.

We assume that $\xi \in (0, 1)$, the first beam has spatial extent $[0, \xi]$, the second beam has spatial extent $[\xi, 1]$, and the beams are hinged at 0 and 1. Both are uniform Euler-Bernoulli beams with the same mass density per unit length m and the same flexural rigidity EI. We normalize so that $EI/m = 1$. Let $w(\zeta, t)$ be the displacement of the coupled beams at position $\zeta \in [0, 1]$ and time $t > 0$. The notation $\dot{w}(\zeta, t)$ denotes the derivative of $w(\zeta, t)$ with respect to time, and D denotes the spatial differentiation operator. Then w satisfies the following equations:

$$\begin{aligned} \ddot{w}(\zeta, t) + D^4 w(\zeta, t) &= f(\zeta) d(t), \qquad \zeta \in (0, \xi) \cup (\xi, 1) \\ w(0, t) = w(1, t) = 0, &\quad D^2 w(0, t) = D^2 w(1, t) = 0, \\ w(\xi^-, t) = w(\xi^+, t), &\quad Dw(\xi^-, t) = Dw(\xi^+, t) \end{aligned} \tag{5.1}$$

where the disturbance $d(t)$ is of the form (4.1) and the spatial distribution of the disturbance is given by the function $f(\zeta)$. We assume for simplicity that $f \in L^2[0, 1]$, but the development will work for some distributions f as well. See the Introduction of [4] for a discussion of the physical meaning of these joint conditions.

In order to solve the disturbance rejection problem for the coupled beams (5.1), we apply a stabilizing feedback, and then apply the controller described in Theorem 1.2, which does not require the gain to be small. It is possible to exponentially stabilize the coupled beams with either a *rigid support* joint (a pointwise shear force feedback) or an *angle guide* joint (a pointwise bending moment feedback); see [4] for a discussion of the physical meaning of these feedback controls, and [25] for exponential stability and well-posedness results. For instance, the closed-loop system with the shear force feedback is exponentially stable when $\xi = 1/2$, and the closed-loop system with bending moment feedback is exponentially stable with $\xi = 1/3$, and either feedback leads to a well-posed closed-loop system for any joint placement. Hence we can apply Theorem 1.1 to obtain a tracking and disturbance rejection to either of these closed-loop systems. Unfortunately, the exponential stability of this system with either of these feedbacks is very non-robust with respect to the joint position (see [25]), hence it is not likely to be practical. Therefore, we will apply a feedback which has been recently shown in Ammari *et al.* [1] to exponentially stabilize in a way which is robust with respect to joint position. In [1] it is shown that if *both* types of dissipative feedback mechanisms are incorporated into the joint, then the system is exponentially stable independent of the joint position ξ. Thus, [1] considers the equations (5.1) together with the joint conditions

$$D^3w(\xi^+,t)-D^3w(\xi^-,t) = -\dot{w}(\xi,t), \quad D^2w(\xi^-,t)-D^2w(\xi^+,t) = -D\dot{w}(\xi,t). \tag{5.2}$$

It is shown in [1] that for any $\xi \in (0,1)$, (5.1)–(5.2) determine an exponentially stable semigroup describing the evolution of $[w(\cdot,t)\ w_t(\cdot,t)]^\top$ in the state space

$$X = \{[x_1\ x_2]^\top \in H^2(0,1) \times L^2(0,1) \mid x_1(0) = x_1(1) = 0\},$$

with inner product $\langle [x_1\ x_2]^\top, [y_1\ y_2]^\top \rangle = \int_0^1 \{D^2x_1(\tau)D^2y_1(\tau) + x_2(\tau)y_2(\tau)\}d\tau$.

We need to reformulate (5.1) with (5.2) as the equations describing the state trajectories of an appropriate well-posed system Σ_p. The first step is to show that a related undamped system Σ_0 is well-posed and described by

$$\dot{x}(t) = A_0x(t) + B_dd(t) + B\tilde{u}(t), \qquad z(t) = B^*_\Lambda x(t), \tag{5.3}$$

with the same state space X, input spaces $W = \mathbb{R}$, $U = \mathbb{R}^2$, and output space $Y = \mathbb{R}^2$, with the operators A_0, B_d and B to be determined shortly. In (5.3), B^*_Λ is the Λ-extension of B^*, defined similarly to (2.5). We will then show that Σ_p is obtained from Σ_0 by negative output feedback, as shown in Figure 5.

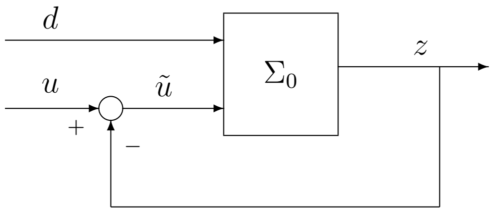

Figure 5: The damped coupled beams system Σ_p, obtained from the undamped coupled beams system Σ_0 by negative output feedback. The transfer function $\mathbf{Q}$ from $\tilde{u}$ to z, and hence the transfer function $\mathbf{P}_2$ from u to z, are positive.

We choose $B_d = [0\ f(\cdot)]^\top$. We use the notation $\tilde{u}$ for the control input of Σ_0. The state $[w\ \dot{w}]^\top$ of Σ_0 satisfies (5.1) and

$$\begin{aligned} D^3w(\xi^+,t) - D^3w(\xi^-,t) &= \tilde{u}_1(t), \qquad D\dot{w}(\xi,t) = \tilde{u}_2(t), \\ z_1(t) = \dot{w}(\xi,t), &\qquad z_2(t) = D^2w(\xi^-,t) - D^2w(\xi^+,t). \end{aligned} \tag{5.4}$$

Let $\tilde{u} = [\tilde{u}_1 \ \tilde{u}_2]^\top$ and $z = [z_1 \ z_2]^\top$. We need to identify the control operator B that corresponds to (5.1) with (5.4). We will use the approach in Section 3 of Ho and Russell [11]. As usual, (5.3) will be true as an equation in $X_{-1} := \mathcal{D}(A_0^*)'$, so A_0 is extended to an operator from X to X_{-1} by

$$\langle A_0 x, v\rangle = \langle x, A_0^* v\rangle \quad \text{for all} \ \ v \in \mathcal{D}(A_0^*) = \mathcal{D}(A_0).$$

We use the same symbol A_0 for this standard extension.

For (5.1) with (5.4) A_0 (and its domain) is given by

$$A_0 := \begin{bmatrix} 0 & I \\ -D^4 & 0 \end{bmatrix}, \tag{5.5}$$

$$\begin{aligned}\mathcal{D}(A_0) = \Big\{ & [x_1 \ x_2]^\top \in X \mid x_1 \in H^4(0,\xi) \cup H^4(\xi,1), \ x_2 \in H^2(0,1), \\ & D^2 x_1(0) = D^2 x_1(1) = 0, \ x_2(0) = x_2(1) = 0, \\ & Dx_2(\xi) = 0, \ D^3 x_1(\xi^+) = D^3 x_1(\xi^-)\Big\}.\end{aligned}$$

The operator D^4 is applied separately on the intervals $(0,\xi)$ and $(\xi,1)$. It is shown in [25, Section 3] that A_0 is a skew-adjoint operator (which has a Riesz basis of eigenvectors) on X, hence it generates a unitary semigroup S on X.

Proposition 5.1 *For* $[v_1 \ v_2]^\top \in \mathcal{D}(A_0^*) = \mathcal{D}(A_0)$, *define*

$$B_1^*[v_1 \ v_2]^\top = v_2(\xi), \qquad B_2^*[v_1 \ v_2]^\top = D^2 v_1(\xi^-) - D^2 v_1(\xi^+),$$

and $B = [B_1 \ B_2]$. *Then* B *is an admissible control operator for* S, B^* *is an admissible observation operator for* S, *and the system* Σ_0 *described by* (5.1) *with* (5.4) *is regular and it can be written in the form* (5.3).

Proof. It is obvious that with the above choice of B_1^* and B_2^*, the observations in (5.4) can be written for $x(t) \in \mathcal{D}(A_0)$ as $z(t) = B^* x(t)$. Let A_1 be the extension of A_0 given by the same matrix of operators (5.5), with the following domain, which no longer requires $Dx_2(\xi) = 0$ and $D^3 x_1(\xi^+) = D^3 x_1(\xi^-)$, since these are boundary conditions in which the controls are incorporated:

$$\begin{aligned}\mathcal{D}(A_1) = \Big\{ & [x_1 \ x_2]^\top \in X \mid x_1 \in H^4(0,\xi) \cup H^4(\xi,1), \ \ x_2 \in H^2(0,1), \\ & D^2 x_1(0) = D^2 x_1(1) = 0, \ \ x_2(0) = x_2(1) = 0\Big\}.\end{aligned}$$

Then (5.1) is equivalent to

$$\dot{x}(t) = A_1 x(t), \qquad x(t) = [w(t) \ \dot{w}(t)]^\top.$$

Using integration by parts, we obtain for $x = [x_1 \ x_2]^\top \in \mathcal{D}(A_1)$ and $v = [v_1 \ v_2]^T \in \mathcal{D}(A_0^*) = \mathcal{D}(A_0)$,

$$\begin{aligned}\langle A_1 x, v\rangle = \langle x, -A_0 v\rangle + [D^3 x_1(\xi^+) - D^3 x_1(\xi^-)]\, v_2(\xi) \\ + Dx_2(\xi)\,[D^2 v_1(\xi^-) - D^2 v_1(\xi^+)]\,.\end{aligned} \tag{5.6}$$

Since $A_0^* = -A_0$, (5.6) implies that the following equation holds in X_{-1}:

$$A_1 x = A_0 x + [D^3 x_1(\xi^+) - D^3 x_1(\xi^-)]B_1 + D x_2(\xi) B_2,$$

for all $x \in \mathcal{D}(A_1)$. Since $D^3 w(\xi^+, t) - D^3 w(\xi^-, t) = \tilde{u}_1(t)$ and $D\dot{w}(\xi, t) = \tilde{u}_2(t)$, (5.1) and the first part of (5.4) can be written as the first part of (5.3).

It is shown in [25, Lemma 3.7] that B_2 is an admissible control operator for S. The admissibility of B_1 requires a standard argument: we will show that the expansion coefficients for B_1 satisfy the Carleson measure criterion, found in [11, Corollary 2.5]. In [25, Proposition 3.3] it is shown that the multiplicity of each eigenvalue of A_0 is one. In the proof of [25, Lemma 3.7] it is shown that for any $h \geq 1$, the number N_h of eigenvalues in $\{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq h,\ a - h \leq \operatorname{Im} z \leq a + h\}$ satisfies $N_h \leq Mh$ for some $M < \infty$ that is independent of $a \in \mathbb{R}$.

Let $\{\Psi_{\pm\omega}\}$ be a complete set of independent eigenvectors of A_0, corresponding to the eigenvalues $\pm i\omega^2$ (described below). Let $C_{\pm\omega} > 0$ be chosen so that $\|C_{\pm\omega}\Psi_{\pm\omega}\| = 1$. A detailed description of $\Psi_{\pm\omega}$ is given in [25, Lemma 3.2], and $C_{\pm\omega}$ is given by equation (3.38) in [25]. Let $b_{\pm\omega} = C_{\pm\omega} B_1^* \Psi_{\pm\omega}$. If $(b_{\pm\omega})$ is a bounded sequence, then according to the Carleson measure criterion B_1 is admissible. The eigenvectors associated with nonzero eigenvalues $\pm i\omega^2$ are of the form

$$\Psi_{\pm\omega} = \begin{bmatrix} \psi_\omega / \pm i\omega^2 \\ \psi_\omega \end{bmatrix} \tag{5.7}$$

with $\psi_\omega \in H^2(0,1)$, so $b_{\pm\omega} = C_{\pm\omega}\psi_\omega(\xi)$.

There are two types of eigenvectors of A_0 associated with non-zero eigenvalues. If $\cos(\pi n \xi) = 0$ for some $n \in \mathbb{Z}^+$, then $\lambda = \pm i(\pi n)^2$ are eigenvalues with associated eigenvectors as in (5.7), with $\omega = \pi n$ and $\psi_\omega(s) = \sin(\pi n s)$. In this case $C_{\pm\omega} = 1$, so $b_{\pm\omega} = \sin(\pi n \xi)$, and these coefficients are clearly bounded.

Let ω be a positive real solution of

$$\tanh(\omega\xi) - \tan(\omega\xi) - \tanh(\omega(1-\xi)) + \tan(\omega(1-\xi)) = 0.$$

Then $\lambda = \pm i\omega^2$ are eigenvalues of A_0 with the associated eigenvectors given in (5.7), with ψ_ω given in part (c) of [25, Lemma 3.2]. We compute that

$$\psi_\omega(\xi) = \tanh(\omega\xi) - \tan(\omega\xi). \tag{5.8}$$

Furthermore, rewriting (3.38) in [25],

$$\frac{1}{C_{\pm\omega}^2} = \xi\left[\tan^2(\omega\xi) + \tanh^2(\omega\xi)\right] + (1-\xi)\left[\tan^2(\omega(\xi-1)) + \tanh^2(\omega(\xi-1))\right].$$

Since $\xi \in (0,1)$, it is clear from this formula and (5.8) that $(b_{\pm\omega}) = (C_{\pm\omega}\psi_\omega(\xi))$ is a bounded sequence for these eigenvectors. This shows that B_1 is an admissible control operator for S, hence B is an admissible control operator for S. By a standard duality argument, B^* is an admissible observation operator for S.

In [25] it is shown that (5.1) and (5.4) with $u_1 = 0$ (i.e., with B_1 replaced by the zero vector) determine a well-posed and regular system with feedthrough operator 0. The proof follows directly from the eigenvalue placement and the boundedness of the sequence of expansion coefficients for B_2 (see [25, Theorem 2.7]), which is also true for B_1, so our system Σ_0 is well-posed and regular. ■

Now we show that the damped system Σ_p with inputs d, u_1 and u_2 and outputs z_1 and z_2, described by (5.1), (5.2) and

$$z_1(t) = \dot{w}(\xi, t), \qquad z_2(t) = D^2 w(\xi^-, t) - D^2 w(\xi^+, t), \tag{5.9}$$

is well-posed. We do this by showing that Σ_p is obtained from Σ_0 by static output feedback. Let $\mathbf{Q}$ be the transfer function from $\tilde{u}$ to z for Σ_0. We claim that $\mathbf{Q}$ is positive. The *degree of unboundedness* of any $B \in \mathcal{L}(U, X_{-1})$, denoted $\alpha(B)$, is the infimum of those $\alpha \geq 0$ for which there exist positive constants δ, ω such that

$$\|(\lambda I - A)^{-1} B\|_{\mathcal{L}(U,X)} \leq \frac{\delta}{\lambda^{1-\alpha}} \qquad \forall\, \lambda \in (\omega, \infty)$$

(this was introduced in [26, Section 1]). It is shown in Curtain and Weiss [5, Remark 5.4] that if a well-posed system has a skew-adjoint generator A_0, a control operator B with $\alpha(B) < 1/2$, and its observation operator is B^*, then this system is regular and the transfer function $\mathbf{Q}(s) = B^*_\Lambda (sI - A)^{-1} B$ is positive. To apply this result, we need to show that $\alpha(B) < 1/2$. For $C_{\pm\omega}$ and $\Psi_{\pm\omega}$ defined above, it is shown in [25, Section 3] that $(\Phi_{\pm\omega}) := (C_{\pm\omega} \Psi_{\pm\omega})$ is a Riesz basis of eigenvectors of A_0 for X. We have shown above that the expansion coeffients for B_1 in this basis are bounded, and in [25, Lemma 3.7] the same is shown for B_2. It is shown in [25, Lemma 3.6] that the eigenvalues of A_0 can be enumerated so that $\lambda_{\pm k} = \pm i\omega_k^2$, where there exists $a, b \in \mathbb{R}$, $m > 0$ such that $mk + b \leq w_k \leq mk + a$ for $k \in \mathbb{N}$. Therefore, it is easy to see that for any $\varepsilon > 0$, we can write $B = A^{1/4+\varepsilon} \tilde{B}$, where $\tilde{B}$ is bounded. It then follows from Krein [12, equation (5.15)] that $\alpha(B) \leq 1/4$, which shows that $\mathbf{Q}$ is positive. Hence $\mathrm{Re}\,(I + \mathbf{Q}(s)) \geq I$ for $s \in \mathbb{C}_0$ and this implies that $K = [0 \ \ -I]^\top$ is an admissible feedback operator for Σ_0 (see Section 2 for this concept). Let $u = [u_1 \ u_2]^\top$. Thus, the feedback $\tilde{u}(t) = -z(t) + u(t)$ for Σ_0 yields the well-posed and regular damped system Σ_p. According to (2.11) and (2.12), Σ_p can be written (using that $[B_d \ B]K = -B$) in the form

$$\dot{x}(t) = (A_0 - BB^*_\Lambda)x(t) + B_d d(t) + Bu(t), \qquad z(t) = B^*_\Lambda x(t). \tag{5.10}$$

Thus, the semigroup generator of Σ_p is $A = A_0 - BB^*_\Lambda$. Using the feedback relation $\tilde{u}(t) = -z(t) + u(t)$ in (5.4), we obtain (5.2) and (5.9), so that (5.1), (5.2) and (5.9) are represented by (5.10). As mentioned earlier, it follows from [1] that A generates an exponentially stable semigroup.

The transfer function of Σ_p is $\mathbf{P} = [\mathbf{P}_1 \ \mathbf{P}_2]$, where $\mathbf{P}_2(s) = \mathbf{Q}(s)(I + \mathbf{Q}(s))^{-1}$, which is easily seen from this formula to be positive. We will apply Theorem 1.2 to this system. We note that since B_d is a bounded operator, the transfer function from d to z is well-posed. Therefore, to reject the disturbance while maintaining exponential stability, we can apply Theorem 1.2 to the system described by (5.1), (5.2) and (5.9) as long as $\{\omega_j\}_{j\in\mathcal{J}} \subset \mathbb{R}$ is such that $\operatorname{Re}\mathbf{P}_2(i\omega_j)$ is invertible for all $j \in \mathcal{J}$. Let us examine this condition in terms of $\mathbf{Q}(i\omega_j)$. Since it is easy to verify that $\mathbf{Q}^*(i\omega_j) = -\mathbf{Q}(i\omega_j)$, we see that $\operatorname{Re}\mathbf{P}_2(i\omega_j) = -\mathbf{Q}(i\omega_j)^2[I - \mathbf{Q}(i\omega_j)^2]^{-1}$. Since $\operatorname{Re}\mathbf{Q}(i\omega_j)^2 \le 0$, we see that $[I - \mathbf{Q}(i\omega_j)^2]^{-1}$ is invertible for all $j \in \mathcal{J}$, so $\operatorname{Re}\mathbf{P}_2(i\omega_j)$ is invertible if and only if $\mathbf{Q}(i\omega_j)$ is invertible.

As an example, we consider the case when $\xi = 1/2$. Write $s = i\eta^2$. In this case

$$\mathbf{Q}(s) = \begin{bmatrix} \mathbf{Q}_{11}(s) & 0 \\ 0 & \mathbf{Q}_{22}(s) \end{bmatrix},$$

where

$$\mathbf{Q}_{11}(s) = \frac{4i}{\eta}\{\tan\frac{\eta}{2} - \tanh\frac{\eta}{2}\}, \qquad \mathbf{Q}_{22}(s) = \frac{4}{i\eta}\frac{\sinh\frac{\eta}{2}\sin\frac{\eta}{2}}{\sin\frac{\eta}{2}\cosh\frac{\eta}{2} - \sinh\frac{\eta}{2}\cos\frac{\eta}{2}}.$$

Thus, $\mathbf{Q}(s)$ is nonsingular everywhere except at points where $\mathbf{Q}_{11}(s) = 0$ or $\mathbf{Q}_{22}(s) = 0$. We have that $\mathbf{Q}_{11}(s) = 0$ when $s = \pm i\eta_{1,k}^2$, where $\{\eta_{1,k}\}_{k=0}^{\infty}$ is asymptotic to $\{\pi/2 + 2\pi k\}_{k=0}^{\infty}$. Furthermore, we can see that $\mathbf{Q}_{22}(s) = 0$ when $s = \pm i\eta_{2,k}^2$, where $\eta_{2,k} = 2\pi + 2\pi k$ for $k \in \mathbb{Z}^+$. Both $\mathbf{Q}_{11}$ and $\mathbf{Q}_{22}$ have removable singularities at 0, and their analytic extensions have a zero at 0. Thus, we can track signals containing components at all frequencies ω except 0 and $\{\pm\eta_{1,k}^2, \pm\eta_{2,k}^2\}_{k=0}^{\infty}$.

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