Boundary Controllability of a Coupled Wave/Kirchoff System

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Abstract

We consider two problems in boundary controllability of coupled wave/Kirchoff systems. Let Ω be a bounded region in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz continuous boundary Γ. In the motivating structural acoustics application, Ω represents an acoustic cavity. Let Γ_0 be a flat subset of Γ which represents a flexible wall of the cavity. Let z denote the acoustic velocity potential, which satisfies a wave equation in Ω, and let v denote the displacement on Γ_0, which satisfies a Kirchoff plate equation on Γ_0. These equations are coupled via $\partial z/\partial \nu = v_t$ on Γ_0 (where ν is the exterior unit normal to Γ_0), and the backpressure $-z_t$ appears in the Kirchoff equation. In the first problem, we consider a control u_0 in the Kirchoff equation on Γ_0, and an additional control u_1 in the Neumann conditions on a subset Γ_1 of Γ, where $\Gamma \setminus \Gamma_1$ satisfies geometric conditions. Using both controls, we obtain exact controllability of the wave and plate components in the natural state space. In the second problem we consider only the control u_0. Without geometric conditions, exact controllability is not possible, but we show that for any initial data, we can steer the plate component exactly, and the wave component approximately.

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1 Introduction

In this paper we consider certain controllability questions for two systems of coupled partial differential equations which model the interaction between sound waves in an acoustic cavity and the walls of the cavity. The first model we study has been considered in [8, 9, 10, 12, 6]. Let Ω be a bounded region with Lipschitz continuous boundary Γ in $\mathbb{R}^n$, for $n \geq 2$. In the motivating structural acoustics application, the region Ω represents an acoustic cavity. Let Γ_0 be a flat subset of Γ which represents a flexible wall of the cavity; this boundary portion is sometimes referred to as the "active" wall of the cavity. Let (the wave component) z denote the acoustic velocity potential, which satisfies the wave equation in Ω with homogeneous Neumann boundary conditions on $\Gamma \setminus \Gamma_0$. Let (the plate component) v denote the displacement on region Γ_0, and which satisfies a Kirchoff equation. The coupling between these two equations can be described roughly as follows: if ν is the exterior unit normal on Γ_0, then $\partial z/\partial \nu = v_t$ on Γ_0; furthermore, the backpressure term $-z_t$ appears in the Kirchoff equation. Several variants of this model have also been studied in the literature, including a model for the case where the active wall is modelled by an Euler-Bernoulli plate (with Kelvin-Voigt damping), rather then by a Kirchoff plate. Structural acoustic systems which have their active wall dynamics modelled by a plate equation are well-motivated and have an extensive literature. In Avalos and Lasiecka [5] and Micu and Zuazua [22] controllability of a similar system is studied, but in the case that the active boundary Γ_0 is modelled by a wave equation.

The goal of this paper is to obtain some notion of controllability in the natural state space, in the physically motivated situation where control is applied to the *boundary* of Ω. In a typical application, control is applied to the active boundary Γ_0, appearing as an inhomogeneous term in the Kirchoff plate equation. In many applications, it would be desirable to have this Γ_0-term as the only control for the system. However, without certain geometric conditions in place, it would be unreasonable to expect exact controllability of the system with controls only on Γ_0; see for example the classic paper by Bardos, *et al.* [7], where it is shown that certain geometric conditions are necessary for exact boundary controllability of the wave equation. Therefore, to obtain exact controllability for the wave component of the dynamics, we introduce an additional control on a subset Γ_1 of Γ, and with $\Gamma \setminus \Gamma_1$ satisfying certain geometric conditions. In this case, we see in Theorem 2 that we can indeed obtain the desired exact controllability of the wave and plate components. This result depends on: (i) having good observability estimates for both wave and plate equations; (ii) correctly handling the coupling between the wave and plate dynamics, which accounts for the major mathematical difficulty and novelty. To reconcile the (unbounded) nature of this coupling we will apply technical "negative norm" estimates for the boundary traces of solutions to second order hyperbolic equations (see (2.23)). It is well known that standard Sobolev trace theory is not applicable–nay, not even valid–on Sobolev spaces of negative scale. Therefore in the work undertaken to obtain the requisite observability estimate, it will be indispensible that we use sharp trace regularity theory which takes advantage of the underlying hyperbolicity and related propagation of singularities.

We still do not want to give up on the idea of controlling on Γ_0 alone. In this case, we wish to prove controllability of the structural acoustic system in a more limited sense. Unfortunately, the approximate controllability of the model described above is problematic, and in fact may not hold for any subspace which allows for an invariance of the flow. Therefore, we consider a slightly modified system which is still well-motivated physically, with Robin boundary conditions on $\Gamma \setminus \Gamma_0$, instead of Neumann conditions (see (2.3) below). With the approximate controllability property being valid for this modified system, we are able to show the following in Theorem 4: For any initial data, we can steer the plate component *exactly*, and the wave component *approximately*. Both of our results depend upon special behavior of the traces which are not obtainable by standard Sobolev trace theory.

There are not many exact boundary controllability results in the literature for coupled systems. For an early result on controllability of a coupled structure, which concerns a beam (partial differential) equation coupled with an ordinary differential equation, see Littman and Marcus [20]. For exact controllability of systems more closely related to those of the present paper, see [5, 22] wherein the plate equation on Γ_0 is replaced by a wave equation. See also [1], which follows very much the methodology of [22] in obtaining an exact controllability property for an interior wave coupled to a Euler-Bernoulli beam model. However, like that in [22], the result in [1] pertains to very special spaces of initial data (much narrower than that of finite energy) and on rectangular domains.

In Section 2 we state and prove Theorem 2, which concerns the case where two controls are used to obtain exact controllability of the coupled system. In Section 3 we state and prove Theorem 4, which handles the case where one control (on Γ_0) only is used, and where no geometric conditions are imposed. In this case, Theorem 4 yields *exact/approximate* controllability of the system.

In this paper C denotes a generic constant. If C depends upon a variable, we include that variable as a subscript for C.

2 Control on $\Gamma_0 \cup \Gamma_1$

We start by describing in detail the first coupled system we are considering. We decompose the boundary $\partial\Omega$ into $\Gamma_0 \cup \Gamma_1 \cup \Gamma_*$, where $\Gamma_0 \neq \emptyset$, $\Gamma_1 \neq \emptyset$ and $\Gamma_0 \cap \Gamma_1 \cap \Gamma_* = \emptyset$. We also introduce the notation $\vec{z} = [z \; z_t]^T$ and $\vec{v} = [v \; v_t]^T$. Let the (rotational inertia) parameter $\gamma > 0$. The following is our structural acoustics model with two controls.

$$\begin{cases} z_{tt} = \Delta z \quad \text{on } (0,T) \times \Omega \\ \dfrac{\partial z}{\partial \nu} = \begin{cases} u_1 \text{ on } (0,T) \times \Gamma_1 \\ v_t \text{ on } (0,T) \times \Gamma_0 \\ 0 \text{ on } (0,T) \times \Gamma_* \end{cases} \\ \\ v_{tt} - \gamma \Delta v_{tt} = -\Delta^2 v + u_0 - z_t|_{\Gamma_0} \quad \text{on } (0,T) \times \Gamma_0 \\ v|_{\partial\Gamma_0} = \Delta v|_{\partial\Gamma_0} = 0 \;\; \text{on } (0,T) \times \partial\Gamma_0 \\ \\ \vec{z}(0) = \vec{z}_0 \in H_1; \;\; \vec{v}(0) = \vec{v}_0 \in H_0, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} H_1 &\equiv H^1(\Omega) \times L^2(\Omega); \\ H_0 &\equiv (H^2(\Gamma_0) \cap H_0^1(\Gamma_0)) \times H_0^1(\Gamma_0). \end{aligned}$$

We will need the following assumptions for the geometry of Ω, the second of which is essentially one of convexity on $\Gamma_0 \cup \Gamma_*$[1]:

(A.1) Γ_0 is flat and Γ_* is convex (in the sense conveyed in the previous footnote);

(A.2) There exists a point $x_0 \in \mathbb{R}^n$ such that

$$(x - x_0) \cdot \nu \leq 0, \quad x \in \Gamma_0 \cup \Gamma_*. \tag{2.2}$$

Remark 1 *If (A.1) and (A.2) are satisfied, then it is possible to construct a vector field $h(x)$ such that $h \cdot \nu = 0$ on $\Gamma_0 \cup \Gamma_*$, and $J(h) > \rho_0 > 0$ on Ω, where $J(h)$ is the Jacobian of h, [13]. If $\Gamma_* = \emptyset$, then the assumption (A.2) can be removed, since in that case one can simply take a point x_0 on a flat manifold containing Γ_0 (so that $(x - x_0) \cdot \nu = 0$ for $x \in \Gamma_0$). In the case when Ω is a rectangular region, (A.2) is satisfied if Γ_0 and Γ_* are adjacent sides of the boundary of Ω.*

In the absence of control; i.e., when $u_0 \equiv 0$ and $u_1 \equiv 0$ in (2.1), it can be readily shown that for initial data $[\vec{z}_0, \vec{v}_0]$ in $H_1 \times H_0$, the solution $[\vec{z}, \vec{v}]$ evolves continuously into $H_1 \times H_0$ and is conservative with respect to the seminorm (see; e.g., [9, 12]).

The main result of this section is the following.

Theorem 2 *Suppose Ω satisfies assumptions (A.1) and (A.2). Then for time*

$$T > 2\sqrt{\operatorname{diam}(\Omega)},$$

the system (2.1) is exactly controllable on the space $H_1 \times H_0$ by means of controls $u_1 \in L^2(0, T; L^2(\Gamma_1))$ and $u_0 \in L^2(0, T; H^{-1}(\Gamma_0))$.

Proof: We first note that the exact controllability of the system (2.1) is equivalent to exact controllability of the following:

$$\begin{cases} z_{tt} = \Delta z \quad \text{on } (0,T) \times \Omega \\ \left[\dfrac{\partial z}{\partial \nu} + z\right]_{\Gamma_1} = u_1^* \text{ on } (0,T) \times \Gamma_1 \\ \dfrac{\partial z}{\partial \nu} = v_t \text{ on } (0,T) \times \Gamma_0 \\ \dfrac{\partial z}{\partial \nu} = 0 \text{ on } (0,T) \times \Gamma_* \\ \\ v_{tt} - \gamma \Delta v_{tt} = -\Delta^2 v + u_0^* - z_t|_{\Gamma_0} \quad \text{on } (0,T) \times \Gamma_0 \\ v|_{\partial\Gamma_0} = \Delta v|_{\partial\Gamma_0} = 0 \text{ on } (0,T) \times \partial\Gamma_0 \\ \\ \vec{z}(0) = \vec{z}_0 \in H_1; \ \vec{v}(0) = \vec{v}_0 \in H_0. \end{cases} \tag{2.3}$$

[1] We say that $\partial\Omega$ is convex if the Hessian of a level set function describing $\partial\Omega$ is strictly positive in the neighborhood of $\partial\Omega$ on the side of Ω.

Indeed, given preassigned target data $[\vec{z}_T, \vec{v}_T] \in H_1 \times H_0$, suppose we can obtain controls $[u_0^*, u_1^*] \in L^2(0,T; H^{-1}(\Gamma_0)) \times L^2(0,T; L^2(\Gamma_1))$ such that the corresponding trajectory $[\vec{z}, \vec{v}]$ of (2.3) satisfies

$$\vec{z}(T) = \vec{z}_T; \quad \vec{v}(T) = \vec{v}_T. \tag{2.4}$$

If the beam velocity component v_t of (2.3) is in $L^2(0,T; L^2(\Gamma_0))$, then by the sharp regularity theory for solutions to wave equations, we will have $z|_{\Gamma_1} \in L^2(0,T; L^2(\Gamma_1))$ (conservatively); see [15], as well as Theorem 3.1 and Theorem 3.3(a)) of [18]. Consequently, this same trajectory $[\vec{z}, \vec{v}]$ will also satisfy the reachability property (2.4) for the original system (2.1), if we take controls $[u_0, u_1]$ in (2.1) to be

$$\begin{aligned} u_1 &\equiv u_1^* - z|_{\Gamma_1} \in L^2(0,T; L^2(\Gamma_1)); \\ u_0 &\equiv u_0^* \in L^2(0,T; H^{-1}(\Gamma_0)) \end{aligned} \tag{2.5}$$

(note that the absolutely necessary regularity $z|_{\Gamma_1} \in L^2(0,T; L^2(\Gamma_1))$ can not be deduced from an appeal to the classical results posted in [21]. Hence sharp regularity for wave equations is of immediate and prime importance in our proof).

Accordingly, provided we can justify that $v_t \in L^2((0,T) \times \Gamma_0)$, it is enough for us to establish the exact controllability property (2.4) of the system (2.3), for given terminal data $[\vec{z}_T, \vec{v}_T] \in H_1 \times H_0$. With this in mind, we proceed to solve the reachability problem (2.3)-(2.4).

Step 1. It is known that the Kirchoff equation on Γ_0 is exactly controllable in arbitrary time $T > 0$ within the class of $L^2(0,T; H^{-1}(\Gamma_0))$-controls (see Theorem 5). Fix $T > 2\sqrt{\text{diam}(\Omega)}$. For given $[\vec{v}_0, \vec{v}_T] \in H_0 \times H_0$, we can thus find a control $\tilde{u} \in L^2(0,T; H^{-1}(\Gamma_0))$ such that the corresponding $[v(\tilde{u})(t)\ v_t(\tilde{u})(t)]^T$ solves the reachability problem

$$\begin{cases} v_{tt} - \gamma\Delta v_{tt} = -\Delta^2 v + \tilde{u} \quad \text{on } (0,T) \times \Gamma_0 \\ v|_{\partial\Gamma_0} = \Delta v|_{\partial\Gamma_0} = 0 \quad \text{on } (0,T) \times \partial\Gamma_0 \\ \vec{v}(0) = \vec{v}_0 \in H_0 \\ \vec{v}(T) = \vec{v}_T. \end{cases} \tag{2.6}$$

Since $\tilde{u} \in L^2(0,T; H^{-1}(\Gamma_0))$, the known regularity theory for the Kirchoff plate – see eg., expression (2.5) of [17] – gives that

$$\vec{v}(\tilde{u}) \in C([0,T]; H_0). \tag{2.7}$$

In particular, $v_t(\tilde{u}) \in C([0,T]; H_0^1(\Gamma_0))$, and so if we extend v_t by zero on $\Gamma \backslash \Gamma_0$, the resulting function $v_t^e(\tilde{u})$, defined on the *entire* boundary Γ by

$$v_t^e(\tilde{u}) \equiv \begin{cases} v_t(\tilde{u}) & \text{on } (0,T) \times \Gamma_0 \\ 0 & \text{on } (0,T) \times (\Gamma \setminus \Gamma_0) \end{cases}, \tag{2.8}$$

is in $C([0,T]; H^1(\Gamma))$. We have created this extension with a view to applying the regularity theory for wave equations in [23].

Step 2. In this step, we concern ourselves with the boundary value problem

$$\begin{cases} z_{tt} = \Delta z \quad \text{on } (0,T)\times\Omega \\ \dfrac{\partial z}{\partial \nu} + z|_{\Gamma_1} = u_1^* \text{ on } (0,T)\times\Gamma_1 \\ \dfrac{\partial z}{\partial \nu} = v_t(\tilde{u}) \text{ on } (0,T)\times\Gamma_0 \\ \dfrac{\partial z}{\partial \nu} = 0 \text{ on } (0,T)\times\Gamma_* \\ \vec{z}(0) = \vec{z}_0; \end{cases} \tag{2.9}$$

where (fixed) $v_t(\tilde{u})$ is the beam velocity component from *Step 1.* In *Step 3*, we will choose the boundary control u_1^* so that the solution $\vec{z}$ of (2.9) satisfies the desired reachability property (2.4). First, however, we have need to model the PDE (2.9) abstractly, to which end we introduce the following operator theoretic machinery:

(i) Let the positive definite, self-adjoint operator $A : L^2(\Omega) \to L^2(\Omega)$ be defined by

$$\begin{aligned} Af &= -\Delta f, \text{ for } f \in D(A), \\ D(A) &= \left\{ f \in H^2(\Omega) \colon \left[\frac{\partial f}{\partial \nu} + f\right]_{\Gamma_1} = 0; \quad \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_0\cup\Gamma_*} = 0 \right\}. \end{aligned}$$

Therewith, we define the operator $A_1 : H_1 \supset D(A_1) \to H_1$ as

$$\begin{aligned} A_1 &\equiv \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \\ D(A_1) &= D(A) \times H^1(\Omega). \end{aligned}$$

If we endow the Hilbert space H_1 with the inner product

$$([z_0, z_1], [\tilde{z}_0, \tilde{z}_1])_{H_1} = \int_\Omega \nabla z_0 \cdot \nabla \tilde{z}_0 d\Omega + \int_{\Gamma_1} z_0|_{\Gamma_1}\, \tilde{z}_0|_{\Gamma_1}\, d\Gamma_1 + \int_\Omega z_1 \tilde{z}_1 d\Omega, \tag{2.10}$$

(with the induced norm being equivalent to that of the usual $H^1(\Omega)\times L^2(\Omega)$-inner product, by elliptic theory), then the Lumer-Phillips Theorem yields that A_1 generates a C_0-group $\{e^{A_1 t}\}_{t\geq 0}$ on H_1.

(ii) For $i = 0, 1$, we define the maps $N_i : L^2(\Gamma_i) \to L^2(\Omega)$ by having

$$\begin{cases} \Delta N_1 g_1 = 0 \quad \text{on } \Omega \\ \dfrac{\partial}{\partial \nu} N_1 g_1 + N_1 g_1 = g_1 \quad \text{on } \Gamma_1 \\ \dfrac{\partial}{\partial \nu} N_1 g_1 = 0 \quad \text{on } \Gamma_0 \cup \Gamma_*, \end{cases} \tag{2.11}$$

$$\begin{cases} \Delta N_0 g_0 = 0 \quad \text{on } \Omega \\ \dfrac{\partial}{\partial \nu} N_0 g_0 + N_0 g_0 = 0 \quad \text{on } \Gamma_1 \\ \dfrac{\partial}{\partial \nu} N_0 g_0 = \begin{cases} g_0 & \text{on } \Gamma_0 \\ 0 & \text{on } \Gamma_* \end{cases}. \end{cases} \tag{2.12}$$

With this operator theoretic machinery, the solution $[z, z_t]$ to the uncoupled wave equation (2.9) has the explicit representation

$$\begin{bmatrix} z(t) \\ z_t(t) \end{bmatrix} = e^{A_1 t}\vec{z}_0 + \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} + \int_0^t e^{A_1(t-s)} \begin{bmatrix} 0 \\ AN_1 u_1^*(s) \end{bmatrix} ds, \tag{2.13}$$

where the term $[w, w_t]$, corresponding to fixed data $v_t^e(\tilde{u}) \in C([0,T]; H^1(\Gamma))$ (of (2.8)), is defined by

$$\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \equiv \int_0^t e^{A_1(t-s)} \begin{bmatrix} 0 \\ AN_0 v_t(\tilde{u})(s) \end{bmatrix} ds. \tag{2.14}$$

A fortiori, this component $[w, w_t]$ solves the wave equation

$$\begin{cases} w_{tt} = \Delta w \quad \text{on } (0,T) \times \Omega \\ \mathcal{B}w = v_t^e(\tilde{u}) \quad \text{on } (0,T) \times \Gamma \\ [w(0), w_t(0)] = 0, \end{cases} \tag{2.15}$$

where the boundary operator $\mathcal{B}$ is defined by

$$\mathcal{B}f \equiv \begin{cases} \dfrac{\partial}{\partial \nu} f + f, \quad \text{on } \Gamma_1 \\ \dfrac{\partial}{\partial \nu} f, \quad \text{on } \Gamma_0 \cup \Gamma_*. \end{cases}$$

As such, we have from Theorem 3 of [23] that $[w, w_t]$ of (2.14) satisfies the estimate

$$\begin{aligned} \|[w, w_t]\|_{C([0,T];H^1(\Omega))} &\leq C_T \left\|v_t^e(\tilde{u})\right\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} \\ &\leq C_T \left\|v_t(\tilde{u})\right\|_{L^2(0,T;H_0^1(\Gamma_0))}. \end{aligned} \tag{2.16}$$

(see also the "sharper" results in [2, 15, 24], whose full strength is not needed in this context). This regularity for $[w, w_t]$ will come into play presently.

Step 3. Here we consider the exact controllability problem associated with (2.9). Namely, for given terminal data $\vec{z}_T \in H_1$, we wish to specify u_1^* such that the corresponding solution $[z, z_t]$ to (2.9) satisfies at terminal time T

$$\begin{bmatrix} z(T) \\ z_t(T) \end{bmatrix} = \vec{z}_T. \tag{2.17}$$

In this connection, it is useful to define the *control→terminal state* map

$$\mathcal{L}_T^{(1)} : D(\mathcal{L}_T^{(1)}) \subset L^2(0,T; L^2(\Gamma_1)) \to H_1,$$

as

$$\mathcal{L}_T^{(1)} g_1 = \int_0^T e^{A_1(t-s)} \begin{bmatrix} 0 \\ AN_1 g_1(s)\, ds \end{bmatrix} ds. \tag{2.18}$$

Given the abstract representation of $[z, z_t]$ in (2.13), we will then have solved the reachability problem (2.17) if we can find u_1^* such that

$$\mathcal{L}_T^{(1)} u_1^* = \vec{z}_T - e^{A_1 T}\vec{z}_0 - \begin{bmatrix} w(T) \\ w_t(T) \end{bmatrix} \tag{2.19}$$

(note that by (2.16), $\vec{w}(T)$ is well-defined as an element of H_1). Since the geometric assumptions (A.1) and (A.2) are in place, the map $\mathcal{L}_T^{(1)}$, as defined in (2.18), is known to be surjective for $T > 2\sqrt{\text{diam}(\Omega)}$ (see Lasiecka *et al.* [19]). Consequently, there exists a $u_1^* \in L^2(0,T;L^2(\Gamma_1))$ which satisfies (2.19). (In other words, the solution $[z, z_t]$ of (2.9), corresponding to control u_1^*, satisfies the terminal condition (2.17).)

Step 4. Finally, we identify the exact controller of the beam component $[v, v_t]$ in (2.3). To this end, we consider first the trace regularity of the wave component $[z, z_t]$ from *Step 3*, bearing in mind the explicit representation given in (2.13).

We define the wave component

$$\begin{bmatrix} \tilde{w}(t) \\ \tilde{w}_t(t) \end{bmatrix} \equiv e^{A_1 t}\vec{z}_0 \tag{2.20}$$

(corresponding to initial data), and recall also the definition of w in (2.14), (corresponding to boundary data $v_t(\tilde{u})$). We have from Theorem 3 of [23] the trace regularity

$$\begin{aligned} \|[\tilde{w}_t|_\Gamma + w_t|_\Gamma]\|^2_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma))} &\leq C_T\left(\|\vec{z}_0\|^2_{H_1} + \|v_t^e(\tilde{u})\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}\right) \\ &\leq C_T\left(\|\vec{z}_0\|^2_{H_1} + \|v_t(\tilde{u})\|_{L^2(0,T;H_0^1(\Gamma_0))}\right) \end{aligned} \tag{2.21}$$

(note that this needed trace regularity is not available from the classic Sobolev trace theorem; see also "sharper" results in [2, 15, 24], whose full strength is not needed in this context). Moreover, if we denote

$$\begin{bmatrix} w^*(t) \\ w_t^*(t) \end{bmatrix} \equiv \int_0^t e^{A_1(t-s)} \begin{bmatrix} 0 \\ AN_1 u_1^*(s) \end{bmatrix} ds \tag{2.22}$$

(so that $[w^*, w_t^*]$ solves the wave equation with Robin boundary data u_1^* on Γ_1, zero Neumann data on $\Gamma_0 \cup \Gamma_*$, and zero initial data), then quoting Theorem A of [16] (see also [15] for the explicit proof) we have that

$$\| w_t^*|_\Gamma \|_{L^2(0,T;H^{-\frac{4}{5}-\epsilon}(\Gamma))} \leq C_T \, \|u_1^*\|_{L^2((0,T)\times\Gamma_1)} \,. \tag{2.23}$$

(Note that this result is not given explicitly in Theorem A of [16]. However, it can be established by considering the remark at the bottom of page 121 of [16], or by following the details of the proof in [15], where it is noted that the loss of L^2-regularity occurs in a hyperbolic sector where the time Fourier coordinates are comparable to the (space) tangential coordinates; and hence the loss of L^2-regularity does not occur in the time variable.)

Keeping in mind the decomposition $\vec{z} = \vec{w} + \overrightarrow{\tilde{w}} + \overrightarrow{w^*}$, we now set

$$u_0^* = \tilde{u} + z_t|_{\Gamma_0} \,, \tag{2.24}$$

where $\tilde{u}$ is the control from *Step 1.* From (2.21) and (2.23), we have that

$$u_0^* \in L^2(0,T;H^{-1}(\Gamma_0)).$$

Moreover, we have from *Step 1* and the form of the control in (2.24) that the corresponding v-component of the solution to (2.3) satisfies

$$\begin{bmatrix} v(T) \\ v_t(T) \end{bmatrix} = \begin{bmatrix} v(\tilde{u})(T) \\ v_t(\tilde{u})(T) \end{bmatrix} = \vec{v}_T.$$

Combining this terminal state along with that in (2.17), we conclude that with the controls $[u_1^*, u_0^*]$ in place, the solution of the system (2.3) is steered to given terminal data $[\vec{z}_T, \vec{v}_T]$.

To obtain now the reachability property (2.4) for the original system (2.1), we use the fact from *Step 1* that the velocity component v_t of the solution $[\vec{z}, \vec{v}]$ of (2.3) is in $C([0,T]; H_0^1(\Gamma_0))$ (see (2.7)). Now, by virtue of the remark below (2.4), we will have the desired reachability property for the original system (2.1), by taking the controls $[u_1, u_0]$ as prescribed in (2.5).

This completes the proof of Theorem 2. □

Remark 3 *We note that the standard regularity results posted in [21] are not sufficient for us to complete* Step 4 *of the proof. Indeed, in order to appeal to these results so as to infer the well-posedness of the wave velocity* $z_t|_{\Gamma_0}$ *for the Neumann problem, one would need the Neumann data to have "1/2" derivative both in time and space.*

3 Control on Γ_0 only

In this section, we will consider the system (2.3) with no control on Γ_1:

$$\begin{cases} z_{tt} = \Delta z \quad \text{on } (0,T) \times \Omega \\ \left[\dfrac{\partial z}{\partial \nu} + z\right]_{\Gamma \setminus \Gamma_0} = 0 \text{ on } (0,T) \times \Gamma \setminus \Gamma_0 \\ \dfrac{\partial z}{\partial \nu} = v_t \ \text{ on } (0,T) \times \Gamma_0 \\ \\ v_{tt} - \gamma \Delta v_{tt} = -\Delta^2 v + u - z_t|_{\Gamma_0} \quad \text{on } (0,T) \times \Gamma_0 \\ v|_{\partial \Gamma_0} = \Delta v|_{\partial \Gamma_0} = 0 \ \text{ on } (0,T) \times \partial \Gamma_0 \\ \\ [\vec{z}(0), \vec{v}(0)]^T = [\vec{z}_0, \vec{v}_0]^T \in H_1 \times H_0, \end{cases} \tag{3.1}$$

so control is exerted only on the region Γ_0. Geometrically, we will only assume that Γ_0 is flat. In this case of controlling on Γ_0 alone, it is natural–in fact, it is necessary–to consider the interior wave under Robin boundary conditions. Indeed, it is known that steady states cannot be (exactly or approximately) controlled under the Neumann boundary conditions, with control on Γ_0 alone. If one wishes to consider the Neumann problem–and is so abandoning the notion of controlling steady states–he or she can proceed to construct an appropriate subspace $\mathcal{X}$ of $H_1 \times H_0$ which is invariant under the dynamics (viz., $[\vec{z}_0, \vec{v}_0]^T \in \mathcal{X} \Rightarrow [\vec{z}, \vec{v}]^T \in C([0,T; \mathcal{X}])$). This construction will entail the introduction of very restrictive compatibility conditions on the initial data. (Note that in particular, the standard compatibility conditions for the wave equation with Neumann boundary conditions–namely

$\big(H^1(\Omega)/\mathbb{R}\big) \times \big(L^2(\Omega)/\mathbb{R}\big)$–will not work here; we will not have semigroup generation on a quotient space.) In order to keep our analysis focused on the main question of controllability, and not on side issues of compatibility, we will consider the physically relevant Robin boindary conditions in (3.1). For this case one can generally not expect exact controllability for the whole system. Rather, we simultaneously obtain exact controllability in the plate component and approximate controllability in the wave component.

Theorem 4 *Let $T > 2\sqrt{\mathrm{diam}(\Omega)}$. Then given initial data $[\vec{z}_0, \vec{v}_0] \in H_1 \times H_0$, terminal data $[\vec{z}_T, \vec{v}_T] \in H$ and arbitrary $\epsilon > 0$, there exists $u \in L^2(0,T; H^{-1}(\Gamma_0))$, such that at terminal time T the corresponding solution $[\vec{z}, \vec{v}]$ to (3.1) satisfies*

$$\begin{aligned} \|\vec{z}(T) - \vec{z}_T\|_{H_1} &< \epsilon; \\ \vec{v}(T) &= \vec{v}_T. \end{aligned}$$

We prove this using the approach employed in [4]. The key element here is establishing the *partial exact controllability* of the plate component.:

Theorem 5 *Given initial data $[\vec{z}_0, \vec{v}_0] \in H_1 \times H_0$ and terminal data $\vec{v}_T \in H_0$, there exists $u \in L^2(0,T; H^{-1}(\Gamma_0))$ such that for terminal time $T > 2\sqrt{\mathrm{diam}(\Omega)}$, the corresponding solution $[\vec{z}, \vec{v}]$ (3.1) satisfies $\vec{v}(T) = \vec{v}_T$.*

Proof: By classical duality theory, to prove Theorem 5 it is enough to establish the associated observability inequality:

$$\left\|\vec{\psi}_0\right\|_{H_0}^2 \le C_T \int_0^T \int_{\Gamma_0} \left[|\psi_t|^2 + \gamma\,|\nabla \psi_t|^2\right] d\Gamma_0 dt, \tag{3.2}$$

where $\vec{\phi} = [\phi, \phi_t]^T$ and $\vec{\psi} = [\psi, \psi_t]^T$ is the solution of the following homogeneous adjoint system:

$$\begin{cases} \phi_{tt} = \Delta \phi \quad \text{on } (0,T) \times \Omega \\ \left[\dfrac{\partial \phi}{\partial \nu} + \phi\right]_{\Gamma \setminus \Gamma_0} = 0 \text{ on } (0,T) \times \Gamma \setminus \Gamma_0 \\ \dfrac{\partial \phi}{\partial \nu} = \psi_t \text{ on } (0,T) \times \Gamma_0 \\ \\ \psi_{tt} - \gamma \Delta \psi_{tt} = -\Delta^2 \psi - \phi_t|_{\Gamma_0} \quad \text{on } (0,T) \times \Gamma_0 \\ \psi|_{\partial \Gamma_0} = \Delta \psi|_{\partial \Gamma_0} = 0 \text{ on } (0,T) \times \partial \Gamma_0 \\ \\ \left[\vec{\phi}(T), \vec{\psi}(T)\right] = \left[\vec{0}, \vec{\psi}_0\right]. \end{cases} \tag{3.3}$$

We break up the proof of (3.2) into four steps.

Step 1. For $t \ge 0$, let

$$\mathcal{E}_\psi(t) := \frac{1}{2} \int_{\Gamma_0} \left(|\Delta \psi(t)|^2 + |\psi_t(t)|^2 + \gamma\,|\nabla \psi_t(t)|^2\right) d\Gamma_0. \tag{3.4}$$

Multiplying both sides of the Kirchoff plate equation in (3.3) by ψ_t, and then integrating in time and space, we obtain for all $0 \leq s, t \leq T$,

$$\mathcal{E}_\psi(\tau) - \mathcal{E}_\psi(s) = -\int_s^\tau \int_{\Gamma_0} (\phi_t|_{\Gamma_0})\psi_t d\Gamma_0 dt. \tag{3.5}$$

Looking now at the right hand side of the equation (3.5), we note that since ϕ solves the wave equation with Neumann data ψ_t on Γ_0, homogeneous Robin conditions on $\Gamma \setminus \Gamma_0$. and zero initial conditions, the trace term $\phi_t|_{\Gamma_0}$ is well-defined with the estimate (see [23], Theorem 3)

$$\int_0^T \left\| \phi_t|_{\Gamma_0} \right\|^2_{H^{-\frac{1}{2}}(\Gamma_0)} dt \leq C_T \int_0^T \|\psi_t\|^2_{H^{\frac{1}{2}}(\Gamma_0)} dt. \tag{3.6}$$

Applying this estimate to the right hand side of (3.5), we obtain

$$\int_s^\tau \int_{\Gamma_0} (\phi_t|_{\Gamma_0})\psi_t d\Gamma_0 dt = \int_s^\tau \left\langle \phi_t|_{\Gamma_0}, \psi_t \right\rangle_{H^{-\frac{1}{2}}(\Gamma_0)\times H^{\frac{1}{2}}(\Gamma_0)} dt$$

$$\leq \frac{1}{2}\int_0^T \left\| \phi_t|_{\Gamma_0} \right\|^2_{H^{-\frac{1}{2}}(\Gamma_0)} dt + \frac{1}{2}\int_0^T \|\psi_t\|^2_{H^{\frac{1}{2}}(\Gamma_0)} dt \leq C_T \int_0^T \|\psi_t\|^2_{H^{\frac{1}{2}}(\Gamma_0)} dt.$$

We conclude that

$$\int_s^\tau \int_{\Gamma_0} (\phi_t|_{\Gamma_0})\psi_t d\Gamma_0 dt = \mathcal{O}\left(\int_0^T \int_{\Gamma_0} \left[|\psi_t|^2 + \gamma \left|\nabla \psi_t\right|^2 \right] d\Gamma_0 dt \right). \tag{3.7}$$

Combining (3.5) and (3.7) yields that for all $0 \leq s, \tau \leq T$,

$$\mathcal{E}_\psi(\tau) - \mathcal{E}_\psi(s) = \mathcal{O}\left(\int_0^T \int_{\Gamma_0} \left[|\psi_t|^2 + \gamma \left|\nabla \psi_t\right|^2 \right] d\Gamma_0 dt \right). \tag{3.8}$$

Step 2. Let $\Sigma_0 := \Gamma_0 \times [0, T]$. Multiplying the Kirchoff equation in (3.3) by ψ, integrating in time and space, and then integrating by parts, we obtain

$$\begin{aligned} &\int_{\Sigma_0} \left[|\psi_t|^2 + \gamma \left|\nabla \psi_t\right|^2 \right] d\Sigma_0 - \int_{\Gamma_0} \psi_t \psi d\Gamma_0 \Big]_0^T - \gamma \int_{\Gamma_0} \nabla\psi_t \cdot \nabla\psi \Big]_0^T \\ = \quad &\int_{\Sigma_0} |\Delta\psi|^2 d\Sigma_0 + \int_{\Sigma_0} (\phi_t|_{\Gamma_0})\psi d\Sigma_0. \end{aligned} \tag{3.9}$$

Note that for $\delta \in (0, 1)$, we have by elliptic theory and [11] that

$$\langle \nabla\psi_t, \nabla\psi \rangle_{L^2(\Gamma_0)} \leq C \left(\|\psi_t\|^2_{H^{1-\delta}(\Gamma_0)} + \|\psi\|^2_{H^{1+\delta}(\Gamma_0)} \right).$$

Using this and (3.9), we obtain

$$\begin{aligned} \int_{\Sigma_0} |\Delta\psi|^2 d\Sigma_0 \quad \leq \quad &\int_{\Sigma_0} \left[|\psi_t|^2 + \gamma \left|\nabla \psi_t\right|^2 \right] d\Sigma_0 \\ &+ C_\gamma \left\| [\psi, \psi_t] \right\|_{C([0,T]; H_0^{1+\delta}(\Gamma_0)\times H_0^{1-\delta}(\Gamma_0))} - \int_{\Sigma_0} \phi_t|_{\Gamma_0} \psi d\Sigma_0. \end{aligned} \tag{3.10}$$

Applying (3.6) to the last term on the right side side of (3.10) yields

$$\int_{\Sigma_0} |\Delta\psi|^2 \, d\Sigma_0 \le C_T \int_{\Sigma_0} \left[|\psi_t|^2 + \gamma \, |\nabla \psi_t|^2 \right] d\Sigma_0 + C_{T,\gamma} \, \|[\psi, \psi_t]\|_{C([0,T]; H_0^{2-\varepsilon}(\Gamma_0) \times H_0^{1-\varepsilon}(\Gamma_0))} , \tag{3.11}$$

for all sufficiently small $\epsilon > 0$.

Step 3. Integrating the expression for $\mathcal{E}_\psi$ in (3.4) from 0 to T and using the estimate (3.11), we obtain

$$\int_0^T \mathcal{E}_\psi(t) dt \le C_T \int_{\Sigma_0} \left[|\psi_t|^2 + \gamma \, |\nabla \psi_t|^2 \right] d\Sigma_0 + C_{T,\gamma} \, \|[\psi, \psi_t]\|_{C([0,T]; H_0^{2-\varepsilon}(\Gamma_0) \times H_0^{1-\varepsilon}(\Gamma_0))} . \tag{3.12}$$

Employing (3.8) now gives

$$T \mathcal{E}_\psi(T) \le C_T \int_{\Sigma_0} \left[|\psi_t|^2 + \gamma \, |\nabla \psi_t|^2 \right] d\Sigma_0 + C_{T,\gamma} \, \|[\psi, \psi_t]\|_{C([0,T]; H_0^{2-\varepsilon}(\Gamma_0) \times H_0^{1-\varepsilon}(\Gamma_0))} ,$$

or

$$\mathcal{E}_\psi(T) \le C_T \int_{\Sigma_0} \left[|\psi_t|^2 + \gamma \, |\nabla \psi_t|^2 \right] d\Sigma_0 + C_{T,\gamma} \, \|[\psi, \psi_t]\|_{C([0,T]; H_0^{2-\varepsilon}(\Gamma_0) \times H_0^{1-\varepsilon}(\Gamma_0))} . \tag{3.13}$$

This is almost the desired estimate (3.2). To remove the lower order terms, one can use (3.13) in a compactness/uniqueness argument, and the underlying approximate controllability of the system–here is where we need $T > 2\sqrt{diam(\Omega)}$; see the proof of Theorem 4 below, and also [14])–to prove the existence of a constant C_T, for $T > 2\sqrt{diam(\Omega)}$, such that

$$\|[\psi, \psi_t]\|_{C([0,T]; H_0^{2-\varepsilon}(\Gamma_0) \times H_0^{1-\varepsilon}(\Gamma_0))} \le C_T \int_{\Sigma_0} \left[|\psi_t|^2 + \gamma \, |\nabla \psi_t|^2 \right] d\Sigma_0. \tag{3.14}$$

Combining (3.13) and (3.14) establishes (3.2), completing the proof of Theorem 5. □

Proof of Theorem 4:

The proof of Theorem 4 draws on the recipe for the "exact–approximate controller", given in [4], p.370. We sketch here the main details.

Step 1. We first need to show that for $T > 2\sqrt{diam(\Omega)}$, the controlled PDE (3.1) is approximately controllable within the class of controls $u \in L^2(0,T; H^{-1}(\Gamma_0))$. Indeed, by the classical duality theory, it is enough to show that if $\psi_t = 0$, where $\left[\vec{\phi}, \vec{\psi}\right]$ is the solution to the adjoint system

$$\begin{cases} \phi_{tt} = \Delta\phi \quad \text{on } (0,T) \times \Omega \\ \left[\dfrac{\partial \phi}{\partial \nu} + \phi \right]_{\Gamma \setminus \Gamma_0} = 0 \text{ on } (0,T) \times \Gamma \setminus \Gamma_0 \\ \dfrac{\partial \phi}{\partial \nu} = \psi_t \text{ on } (0,T) \times \Gamma_0 \\ \\ \psi_{tt} - \gamma \Delta \psi_{tt} = -\Delta^2 \psi + \phi_t|_{\Gamma_0} \quad \text{on } (0,T) \times \Gamma_0 \\ \psi|_{\partial\Gamma_0} = \Delta\psi|_{\partial\Gamma_0} = 0 \text{ on } (0,T) \times \partial\Gamma_0 \\ \\ \vec{\phi}(T) = \vec{\phi}_0 \in H_1, \ \vec{\psi}(T) = \vec{\psi}_0 \in H_0, \end{cases} \tag{3.15}$$

then necessarily $\vec{\phi}_0 = \vec{0}$ and $\vec{\psi}_0 = \vec{0}$.

To this end, if $\psi_t = 0$, then differentiating the Kirchoff equation in (3.15) yields $\phi_{tt}|_{\Gamma_0} = 0$. Setting $w = \phi_{tt}$, we get from the wave equation in (3.15) that w solves

$$\begin{aligned} w_{tt} &= \Delta w \text{ on } (0,T)\times\Omega \\ \frac{\partial w}{\partial \nu} + w &= 0 \text{ on } (0,T)\times\Gamma\setminus\Gamma_0 \\ \frac{\partial w}{\partial \nu} &= 0 \text{ on } (0,T)\times\Gamma_0 \\ w &= 0 \text{ on } (0,T)\times\Gamma_0. \end{aligned}$$

From Holmgren's Uniqueness Theorem we have that for $T > 2\sqrt{\mathrm{diam}(\Omega)}$, $\phi_{tt} = w = 0$ on $(0,T)\times\Omega$. Again using the wave equation in (3.15), along with elliptic theory and the fact that $\left[\frac{\partial\phi}{\partial\nu} + \phi\right]_{\Gamma\setminus\Gamma_0} = 0$ and $\left.\frac{\partial\phi}{\partial\nu}\right|_{\Gamma_0} = 0$, we get that $\phi = \phi_t = 0$ on $(0,T)\times\Omega$. Finally, using the plate equation in (3.15) we see that since both $\psi_t = 0$ and $\phi_t|_{\Gamma_0} = 0$ on $(0,T)\times\Gamma_0$, $\Delta^2\psi$ is also zero on $(0,T)\times\Gamma_0$. By elliptic theory we then get that $\psi = 0$ on $(0,T)\times\Gamma_0$. So (3.1) is approximately controllable.

Step 2. We define the "control to terminal state map" $\mathfrak{L}_T : L^2(0,T;H^{-1}(\Gamma_0)) \to H_1 \times H_0$, by

$$\mathfrak{L}_T u \equiv \begin{bmatrix} \vec{z}(T) \\ \vec{v}(T) \end{bmatrix}. \tag{3.16}$$

By making use of the regularity theory for Kirchoff plates in [17], and the "decoupling procedure" invoked in [3]–which makes use of the sharp regularity theory (interior *and* trace) for solutions to wave equations under the influence of Neumann boundary data (see [2], [15] and [23])–one can work out that $\mathfrak{L}_T \in \mathfrak{L}\left(L^2(0,T;H^{-1}(\Gamma_0)), H_1\times H_0\right)$. Moreover, we define the projection $\Pi : H_1 \times H_0 \to H_0$ by

$$\Pi \begin{bmatrix} \vec{z}_0 \\ \vec{v}_0 \end{bmatrix} = \vec{v}_0.$$

Given the observability inequality (3.2), one can proceed as in Appendix B of [14] to establish that $\Pi\mathfrak{L}_T\mathfrak{L}_T^*\Pi^* \in \mathfrak{L}(H_0)$ is an isomorphism. Consequently, we will have that

$$\mathfrak{L}_T\mathfrak{L}_T^*\Pi^*\left(\Pi\mathfrak{L}_T\mathfrak{L}_T^*\Pi^*\right)^{-1}\Pi \in \mathfrak{L}(H_1\times H_0). \tag{3.17}$$

Step 3. Let $\left\{e^{\mathcal{A}_t}\right\}_{t\geq 0} \subset \mathfrak{L}(H_1\times H_0)$ be the C_0-semigroup associated with the (homogeneous) dynamics in (3.1), corresponding to generator $\mathcal{A} : D(\mathcal{A}) \subset H_1\times H_0 \to H_1\times H_0$. That is, the solution of (3.1) with $u = 0$ is given by

$$\begin{bmatrix} \vec{z}(t) \\ \vec{v}(t) \end{bmatrix} = e^{\mathcal{A}t}\begin{bmatrix} \vec{z}_0 \\ \vec{v}_0 \end{bmatrix}.$$

Therewith, for arbitrary $\epsilon > 0$, let $u^{(1)} \in L^2(0,T;H^{-1}(\Gamma_0))$ be such that the corresponding solution $\left[\vec{z}^{(1)}, \vec{v}^{(1)}\right]$ satisfies

$$\begin{aligned} & \left\| \begin{bmatrix} \vec{z}^{(1)}(T) - \vec{z}_T \\ \vec{v}^{(1)}(T) - \vec{v}_T \end{bmatrix} + e^{\mathcal{A}T} \begin{bmatrix} \vec{z}_0 \\ \vec{v}_0 \end{bmatrix} \right\|_{H_1 \times H_0} \\ \leq \; & \frac{\epsilon}{1 + \left\| (\mathbb{I} - \Pi^* \Pi) \mathfrak{L}_T \mathfrak{L}_T^* \Pi^* \left(\Pi \mathfrak{L}_T \mathfrak{L}_T^* \Pi^* \right)^{-1} \Pi \right\|_{\mathcal{L}(H_1 \times H_0)}}, \end{aligned} \tag{3.18}$$

where $\mathbb{I}$ is the identity mapping on $H_1 \times H_0$ (since (3.1) is approximately controllable with $L^2(0,T;H^{-1}(\Gamma_0))$-controls, this estimate is surely possible).

Step 4. Let $u^{(2)}$ be the "minimal norm steering control" which steers initial data $[\vec{z}_0, \vec{v}_0]$ in (3.1) to the (partial) target state $\left[\vec{v}^{(1)}(T) - \vec{v}_T\right]$ (by Theorem 5, it makes sense to speak $u^{(2)}$). That is, $u^{(2)}$ minimizes the $L^2(0,T;H^{-1}(\Omega))$-norm over all controls which steer $[\vec{z}_0, \vec{v}_0]$ to $\left[\vec{v}^{(1)}(T) - \vec{v}_T\right]$. By convex optimization (see; e.g., (B.20) of [14]), the minimizer $u^{(2)}$ can be written explicitly as

$$u^{(2)} = \mathfrak{L}_T^* \Pi^* \left(\Pi \mathfrak{L}_T \mathfrak{L}_T^* \Pi^* \right)^{-1} \Pi \left(\begin{bmatrix} \vec{z}_T - \vec{z}^{(1)}(T) \\ \vec{v}_T - \vec{v}^{(1)}(T) \end{bmatrix} - e^{\mathcal{A}T} \begin{bmatrix} \vec{z}_0 \\ \vec{v}_0 \end{bmatrix} \right).$$

Combining this representation with (3.18) gives the estimate

$$\left\| (\mathbb{I} - \Pi^* \Pi) \mathfrak{L}_T u^{(2)} \right\|_{H_1 \times H_0} \leq \epsilon \cdot \frac{\left\| (\mathbb{I} - \Pi^* \Pi) \mathfrak{L}_T \mathfrak{L}_T^* \Pi^* \left(\Pi \mathfrak{L}_T \mathfrak{L}_T^* \Pi^* \right)^{-1} \Pi \right\|_{\mathcal{L}(H_1 \times H_0)}}{1 + \left\| (\mathbb{I} - \Pi^* \Pi) \mathfrak{L}_T \mathfrak{L}_T^* \Pi^* \left(\Pi \mathfrak{L}_T \mathfrak{L}_T^* \Pi^* \right)^{-1} \Pi \right\|_{\mathcal{L}(H_1 \times H_0)}}. \tag{3.19}$$

Step 5. Set the control $u = u^{(1)} + u^{(2)}$. Then mindful of the definitions of controllability map $\mathfrak{L}_T$ and semigroup $\left\{ e^{\mathcal{A}t} \right\}_{t \geq 0}$, we have that the corresponding solution $[\vec{z}(t), \vec{v}(t)]$ of (3.1) satisfies at terminal time T,

$$\begin{aligned} \begin{bmatrix} \vec{z}(T) \\ \vec{v}(T) \end{bmatrix} &= \mathfrak{L}_T u^{(1)} + \mathfrak{L}_T u^{(2)} + e^{\mathcal{A}T} \begin{bmatrix} \vec{z}_0 \\ \vec{v}_0 \end{bmatrix} \\ &= \begin{bmatrix} \vec{z}^{(1)}(T) \\ \vec{v}_T \end{bmatrix} + (\mathbb{I} - \Pi^* \Pi) \left(\mathfrak{L}_T u^{(2)} + e^{\mathcal{A}T} \begin{bmatrix} \vec{z}_0 \\ \vec{v}_0 \end{bmatrix} \right). \end{aligned}$$

From this expression, we have then that $\vec{v}(T) = \vec{v}_T$. Moreover, from steps (3.18) and (3.19), we have

$$\begin{aligned} \| \vec{z}(T) - \vec{z}_T \|_{H_1} &\leq \left\| \vec{z}^{(1)}(T) - \vec{z}_T + (\mathbb{I} - \Pi^* \Pi) e^{\mathcal{A}T} \begin{bmatrix} \vec{z}_0 \\ \vec{v}_0 \end{bmatrix} \right\|_{H_1} + \left\| (\mathbb{I} - \Pi^* \Pi) \mathfrak{L}_T u^{(2)} \right\|_{H_1 \times H_0} \\ &\leq \epsilon. \end{aligned}$$

This completes the proof of Theorem 4.

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