Robustness with respect to sampling for stabilization of Riesz spectral systems*

Richard Rebarber†
Department of Mathematics,
University of Nebraska-Lincoln,
Lincoln, NE 68588-0130, U.S.A.
Email: `rrebarbe@math.unl.edu`

Stuart Townley
School of Engineering, Computer Science and Mathematics,
Department of Mathematical Sciences,
University of Exeter,
Exeter, EX4 4QE, U.K.
Email: `townley@maths.ex.ac.uk`

Abstract

We suppose that a continuous-time feedback is input-output stabilizing for an infinite-dimensional system. We address the question of whether the sampled-data controller obtained by applying idealized sample-and-hold to this continuous-time feedback is also input-output stabilizing if the sampling time is small enough. This question has been previously addressed for fairly general systems under various conditions. In this paper we restrict our attention to Riesz spectral systems, for which we generalize the existing results. Specifically, we give two relatively simple conditions which, combined, are sufficient for the sampled-data controller to be stabilizing. The first condition is a spectrum decomposition for the open-loop system generator, which by itself is necessary, but not sufficient, for the system to be stabilizable by sampled-data control. The second is a summability condition relating the real part of the spectrum of the generator and the expansion coefficients for the input and feedback operators.

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† Corresponding author

1. Introduction

In this paper we consider systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t) \in X, \quad u(t) \in \mathbb{C} \\ y(t) &= Fx(t), \quad y(t) \in \mathbb{C} \end{aligned} \tag{1.1}$$

where X is a complex and separable Hilbert space. We assume that A is the (possibly unbounded) generator of a strongly continuous semigroup $T(t), t \geq 0$ on X, with eigenvalues λ_k, $k = 1, 2, \ldots$, and a Riesz basis of associated eigenvectors φ_k. For Hilbert spaces X and Y, let $\mathcal{B}(X, Y)$ denote bounded operators from X into Y. We assume that $B \in \mathcal{B}(\mathbb{C}, X_{-1})$ and $F \in \mathcal{B}(X, \mathbb{C})$. Here X_{-1} is the completion of X with respect to the norm $\|\cdot\|_{-1}$, where $\|x\|_{-1} := \|(A-\rho)^{-1}x\|$, for any ρ in the resolvent set of A. We define the scalars f_k and b_k via

$$f_k = F\varphi_k \quad \text{and} \quad B = \sum_k b_k \varphi_k \,. \tag{1.2}$$

Without loss of generality we can assume that $X = l^2(\mathbb{C})$ with index set $\mathbb{N}$. The condition that $B \in \mathcal{B}(\mathbb{C}, X_{-1})$ is very general and here is is equivalent to

$$\sum_{n=1}^{\infty} \frac{|b_k|}{1 + |\lambda_k|}^{2} < \infty.$$

In Remark 2.5 we discuss the generalization of our results to systems where $\mathbb{C}$ is replaced by $\mathbb{C}^m$, but for the sake of clarity of exposition the proofs are given in the single-input, single-output case.

In Logemann et al. [3] and Rebarber and Townley [5] we considered a natural question concerning sampled-data stabilization of infinite-dimensional systems: if unity output feedback $u = y$ is stabilizing for the continuous time system (1.1), is its digital implementation also stabilizing? More precisely, is the idealized sample and hold feedback

$$u(t) = y(n\tau) \qquad \text{for} \quad t \in [n\tau, (n+1)\tau) \,, \tag{1.3}$$

stabilizing for (1.1) if $\tau > 0$, the sampling period, is small enough? This was shown in Chen and Francis [2] to be true when X is finite-dimensional, and would seem reasonable for infinite-dimensional systems. In [3], we showed that this is true for two large classes of infinite-dimensional systems. For one class we allow arbitrary generators A, but require B to be bounded and F to be compact; in another class we allow B to be highly unbounded but require A to generate an analytic semigroup and F to be compact. In the context of the single-input single-output Riesz spectral systems as described above:

- systems in the first class have $\{b_k\} \in l^2$ and $\{f_k\} \in l^2$, which in turn implies that

$$\sum_{k=1}^{\infty} |b_k f_k| < \infty \,; \tag{1.4}$$

- systems in the second class have

$$\lambda_k = O(|\text{Re}\lambda_k|) \quad (T(t) \text{ is analytic}) \tag{1.5}$$

If either (1.4) or (1.5) hold, then

$$\sum_k \frac{|b_k f_k|}{1 + |\text{Re}\,\lambda_k|} < \infty\,. \tag{1.6}$$

When X is infinite-dimensional, there are many systems of the form (1.1) which can be stabilized by $u = y$ but cannot be stabilized by sampled-data control. A result from Townley et al. [6] states that if a system can be stabilized by sampled-data control (even in open loop), then a number of restrictive necessary conditions must hold. In particular the operator A must satisfy a spectral decomposition property and additionally have only a finite-dimensional unstable part. In the context of the systems under consideration in this paper, assuming without loss of generality that $\{\text{Re}\ \lambda_k\}$ is nondecreasing, this means that we can find $\beta > 0$ and $\kappa \in \mathbb{N}$ so that.

$$\text{Re}\,\lambda_k < -\beta \quad \text{for all} \quad k \geq \kappa \qquad \text{and} \qquad \text{Re}\,\lambda_k \geq 0, \qquad \text{for all} \quad k < \kappa. \tag{1.7}$$

We show that (1.6) and (1.7) are sufficient conditions for (1.3) to be stabilizing if the sampling period is small enough. In [1] we have an example where (1.6) fails, but nevertheless (1.3) is stabilizing if the sampling period is small enough, which shows that (1.6) is not a necessary condition. Note that in general (1.7) is not a sufficient condition for (1.3) to stabilize (1.1) for all sufficiently small $\tau > 0$, as the counter-example in [5] demonstrates.

2. Stability Result

We will focus on a spectral/frequency domain approach, and therefore at first we ignore the issue of whether the closed-loop feedback operator $A + BF$ generates a C_0-semigroup, and initially state the problem in input output terms. With this input-output approach in mind we define the continuous time open-loop transfer function

$$G(s) := F(sI - A)^{-1}B\,. \tag{2.1}$$

In the Riesz spectral case we are considering,

$$G(s) = \sum_{k=1}^{\infty} \frac{b_k f_k}{s - \lambda_k}.$$

Let

$$x_n := x(n\tau)\,.$$

Applying (1.3) to (1.1) results in the discrete time system

$$x_{n+1} = T(\tau)x_n + \left(\int_0^\tau T(s)B\,ds\right) F x_n \tag{2.2}$$

which in the case that A is invertible (that is, $\lambda_k \neq 0$ for all k), leads to a discrete-time open loop transfer function

$$H_\tau(z) := F(zI - T(\tau))^{-1}(T(\tau) - I)A^{-1}B\,; \tag{2.3}$$

see [3], [4] or [6] for details. In the Riesz spectral case

$$H_\tau(z) = \sum_{k=1}^{\infty} \frac{b_k f_k}{\lambda_k} \left(\frac{e^{\lambda_k \tau} - 1}{z - e^{\lambda_k \tau}}\right).$$

If some $\lambda_k = 0$, we replace the corresponding term in the sum above with $b_k f_k \tau/(z-1)$, which is sufficiently easy to deal with that we assume without loss of generality that all $\lambda_k \neq 0$.

Our continuous time stability assumption is that the closed-loop continuous-time system is input-output stable in the sense that there exists $\epsilon^* \in (0,1)$ so that

$$|1 - G(s)| \geq \epsilon^* \qquad \text{for all} \quad \operatorname{Re} s \geq 0\,. \tag{2.4}$$

This implies that the continuous time closed-loop transfer function is input-output stable. In [3, 4, 6] we showed that power stability of the discrete-time system (2.2) is equivalent to the exponential stability of the sampled data system (1.1) and (1.3). The discrete-time system is input-output stable if there exists $\epsilon \in (0,1)$ such that

$$|1 - H_\tau(z)| > \epsilon \qquad \text{for all} \quad |z| \geq 1\,. \tag{2.5}$$

This is what we prove, although in Corollary 2.3 we translate this to state space stability.

Theorem 2.1 *Suppose that unity output feedback is input-output stabilizing for (1.1), by which we mean there exists $\epsilon^* \in (0,1)$ so that (2.4) holds. If (1.6) and (1.7) hold, then for all $\epsilon \in (0, \epsilon^*)$, there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$ (2.5) holds.*

Proof: We establish (2.5) by using (2.4) and an approximation argument. We divide the proof up into a number of steps. In steps 1 and 2 we look at the infinite-dimensional tails in the transfer functions $G(s)$ and $H_\tau(z)$ and show that they are small in some appropriate sense. In step 3 we look at the finite-dimensional truncation

$$1 - \sum_{k=1}^{K_2 - 1} b_k f_k \frac{e^{\lambda_k \tau} - 1}{\lambda_k (z - e^{\lambda_k \tau})}$$

for $|z| \geq 1$ but sufficiently close to 1. In step 4 we look at the same truncated transfer function, but this time bounded away from 1. In both steps 3 and 4 we rely heavily on a comparison between truncations of the continuous- and discrete-time transfer functions $G(s)$ and $H_\tau(e^{s\tau})$. Step 5 pulls steps 1-4 together.

Step 1. We consider the infinite-dimensional tail in the continuous time transfer function $G(s)$. If $s \in \mathbb{C}_0 := \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ and $K \geq \kappa$, then using (1.7),

$$\sum_{k=K}^{\infty} \left| \frac{b_k f_k}{s - \lambda_k} \right| \leq \sum_{k=K}^{\infty} \left| \frac{b_k f_k}{\operatorname{Re} \lambda_k} \right| .$$

Now using (1.6) it follows easily that there exists a large enough $K_1 \geq \kappa$ so that for all $K \geq K_1$

$$\sup_{s \in \mathbb{C}_0} \sum_{k=K}^{\infty} \left| \frac{b_k f_k}{s - \lambda_k} \right| < \frac{\epsilon^* - \epsilon}{2} . \tag{2.6}$$

Step 2. Next we similarly consider the infinite-dimensional tail in the discrete-time transfer function $H_\tau(z)$. We claim that there exists $K_2 > K_1 \geq \kappa$ such that for all $z \in \mathbb{C}$ with $|z| \geq 1$,

$$\sum_{k=K_2}^{\infty} \left| \frac{b_k f_k}{\lambda_k} \left(\frac{e^{\lambda_k \tau} - 1}{z - e^{\lambda_k \tau}} \right) \right| < \frac{\epsilon^* - \epsilon}{4} . \tag{2.7}$$

To see this we first rearrange the summand in (2.7) so that

$$\sum_{k=K_2}^{\infty} \left| \frac{b_k f_k}{\lambda_k} \left(\frac{e^{\lambda_k \tau} - 1}{z - e^{\lambda_k \tau}} \right) \right| = \sum_{k=K_2}^{\infty} \left| \frac{b_k f_k}{\operatorname{Re} \lambda_k} \left(\frac{\frac{e^{\lambda_k \tau} - 1}{\lambda_k \tau}}{\frac{z - e^{\lambda_k \tau}}{\operatorname{Re} \lambda_k \tau}} \right) \right| . \tag{2.8}$$

Looking at the summand in the right hand side of (2.8), remembering that $\operatorname{Re} \lambda_k < 0$ for $k \geq \kappa$, and using $|z| \geq 1$, we see that

$$\left| \frac{\frac{e^{\lambda_k \tau} - 1}{\lambda_k \tau}}{\frac{z - e^{\lambda_k \tau}}{\operatorname{Re} \lambda_k \tau}} \right| \leq \frac{\left| \frac{e^{\lambda_k \tau} - 1}{\lambda_k \tau} \right|}{\frac{1 - e^{\operatorname{Re} \lambda_k \tau}}{|\operatorname{Re} \lambda_k \tau|}} .$$

Let $\delta > 0$. We need to consider two cases:

(i) If $-\delta < \operatorname{Re}(\lambda_k \tau) < 0$, then taking a Taylor series expansions yields

$$\left| \frac{e^{\lambda_k \tau} - 1}{\lambda_k \tau} \right| = O(1), \qquad \frac{1 - e^{\operatorname{Re} \lambda_k \tau}}{|\operatorname{Re} \lambda_k \tau|} = 1 + O(\delta) ;$$

(ii) If $\operatorname{Re}(\lambda_k \tau) < -\delta$, then

$$\left| \frac{\left(\frac{e^{\lambda_k \tau} - 1}{\lambda_k} \right)}{\left(\frac{z - e^{\lambda_k \tau}}{\operatorname{Re} \lambda_k} \right)} \right| < \frac{2}{1 - e^{-\delta}} \left| \frac{\operatorname{Re} \lambda_k}{\lambda_k} \right| \leq \frac{2}{1 - e^{-\delta}} .$$

Combining cases (i) and (ii) with (1.6) yields (2.7).

Step 3. In this step we consider

$$1-\sum_{k=1}^{K_2-1} b_k f_k \frac{e^{\lambda_k \tau}-1}{\lambda_k(z-e^{\lambda_k \tau})}$$

for those z with $|z| \geq 1$ which are sufficiently close to $z=1$ by making a comparison to the corresponding continuous-time truncated transfer function

$$1-\sum_{k=1}^{K_2-1} \frac{b_k f_k}{s-\lambda_k}\,.$$

To do this, we use a parametrization of z in terms of s and τ, (as in [3]). Indeed, using the analyticity of e^z and the open mapping theorem we can write such z as $z=e^{s\tau}$, with $\operatorname{Re} s>0$, $|s\tau|<\eta$ and $\eta>0$ sufficiently small. We accordingly define

$$S=\{z=e^{s\tau} \mid \text{Re } s>0,\ |s\tau|<\eta\}$$

and then compare the truncated discrete-time function evaluated at $z \in S$ to the continuous-time transfer function evaluated at the corresponding s.

We first write

$$\left|1-\sum_{k=1}^{K_2-1} b_k f_k \frac{e^{\lambda_k \tau}-1}{\lambda_k(z-e^{\lambda_k \tau})}\right| = \left|1-\sum_{k=1}^{K_2-1} \frac{b_k f_k}{s-\lambda_k}+\sum_{k=1}^{K_2-1} b_k f_k \left(\frac{1}{s-\lambda_k}-\frac{e^{\lambda_k \tau}-1}{\lambda_k(z-e^{\lambda_k \tau})}\right)\right| \tag{2.9}$$

and rearrange the third term on the right-hand side to give

$$\sum_{k=1}^{K_2-1} b_k f_k \left(\frac{1}{s-\lambda_k}-\frac{e^{\lambda_k \tau}-1}{\lambda_k(z-e^{\lambda_k \tau})}\right) = \sum_{k=1}^{K_2-1} \frac{b_k f_k}{s-\lambda_k}\left(1-\frac{\frac{e^{\lambda_k \tau}-1}{\lambda_k \tau}}{\frac{e^{s\tau}-e^{\lambda_k \tau}}{(s-\lambda_k)\tau}}\right). \tag{2.10}$$

Now for sufficiently small τ and η we have

$$1-\frac{\frac{e^{\lambda_k \tau}-1}{\lambda_k \tau}}{\frac{e^{s\tau}-e^{\lambda_k \tau}}{(s-\lambda_k)\tau}} = 1-\frac{\frac{e^{\lambda_k \tau}-1}{\lambda_k \tau}}{\frac{(e^{(\lambda_k-s)\tau}-1)e^{s\tau}}{(\lambda_k-s)\tau}} = 1-\frac{1+O(\lambda_k \tau)}{[1+O((s-\lambda_k)\tau)][1+O(s\tau)]}\,.$$

Here we used the fact that $\lambda_k \tau$ is small for all small enough τ, since we have only finitely many λ_k, and that from the definition of S we have $|s\tau|<\eta$. It follows that on S we have that

$$\lim_{(\tau,\eta)\to 0}\left(1-\frac{\frac{e^{\lambda_k \tau}-1}{\lambda_k \tau}}{\frac{e^{s\tau}-e^{\lambda_k \tau}}{(s-\lambda_k)\tau}}\right)=0\,. \tag{2.11}$$

It would now seem reasonable to conclude that the truncated discrete-time transfer function approximates the corresponding truncated continuous-time transfer function arbirarily closely for all $z \in S$ and all sufficiently small τ and η. There is one delicate issue: for $z \in S$ the corresponding s could take values arbitrarily close to the unstable eigenvalues λ_k, where the transfer functions approach arbitrarily large values. Obviously, if no λ_k is unstable, i.e. $\kappa = 1$, then we are done. We assume therefore that $\kappa > 1$ and divide the subsequent analysis into two cases:

Step 3a We first consider those $z \in S$, so that the corresponding s is close to one of the unstable λ_k, i.e. for $k = 1, \ldots, \kappa - 1$. Let this be λ_{k^*} and suppose, specifically, that $|s - \lambda_{k^*}| < a$, where $a \in (0, \alpha)$ and $\alpha > 0$ is the minimum separation of the unstable λ_k. Then

$$\left|\frac{b_{k^*} f_{k^*}}{s - \lambda_{k^*}}\right| \geq \frac{|b_{k^*} f_{k^*}|}{a}\,.$$

If the mode λ_{k^*} is uncontrollable or unobservable, then this term is zero and so it can be ignored. If it is controllable and observable, then it is large. In this latter case, using the argument above that

$$\frac{1}{s - \lambda_k} \approx \left(\frac{e^{\lambda_k \tau} - 1}{\lambda_k (z - e^{\lambda_k \tau})}\right)\,,$$

it follows that by choosing a, τ and η small enough, we can ensure that for some $\mu > \epsilon^*$,

$$\left|1 - b_{k^*} f_{k^*} \frac{e^{\lambda_{k^*} \tau} - 1}{\lambda_{k^*}(z - e^{\lambda_{k^*} \tau})} - \sum_{k=1, k \neq k^*}^{K_2 - 1} b_k f_k \frac{e^{\lambda_k \tau} - 1}{\lambda_k (z - e^{\lambda_k \tau})}\right| \geq \mu > \epsilon^*\,. \tag{2.12}$$

Step 3b If $z \in S$, but the corresponding s satisfies $|s - \lambda_k| \geq a$ for $k = 1, \ldots \kappa - 1$, then all of the summands in

$$\sum_{k=1}^{K_2 - 1} \frac{b_k f_k}{s - \lambda_k}$$

are bounded. Using (2.9), (2.10) and (2.11) it then follows that for all small enough τ and η

$$\left|1 - \sum_{k=1}^{K_2 - 1} b_k f_k \frac{e^{\lambda_k \tau} - 1}{\lambda_k (z - e^{\lambda_k \tau})}\right| \geq \left|1 - \sum_{k=1}^{K_2 - 1} \frac{b_k f_k}{s - \lambda_k}\right| - \frac{\epsilon^* - \epsilon}{4}\,. \tag{2.13}$$

Therefore, combining (2.13) with (2.4), (2.6) and (2.7), we have using $K_2 \geq K_1 \geq \kappa$ that

$$|1 - H_\tau(z)| \overset{(2.7)}{\geq} \left|1 - \sum_{k=1}^{K_2 - 1} b_k f_k \frac{e^{\lambda_k \tau} - 1}{\lambda_k (z - e^{\lambda_k \tau})}\right| - \frac{\epsilon^* - \epsilon}{4} \overset{(2.4, 2.6, 2.13)}{\geq} \epsilon^* - \frac{\epsilon^* - \epsilon}{2} - \frac{\epsilon^* - \epsilon}{4} - \frac{\epsilon^* - \epsilon}{4} = \epsilon\,. \tag{2.14}$$

Step 4. All that remains is to consider those z, with $|z| \geq 1$, that are bounded away from 1. More precisely, choose γ sufficiently small so that if $|z| \geq 1$ and $|z - 1| < \gamma$, then $z \in S$, where S has been fixed by the choices of sufficiently small τ and η in step 3.

Now consider those z with $|z| \geq 1$ and $|z-1| \geq \gamma$. We have that

$$1 - H_\tau(z) = 1 - \sum_{k=0}^{K-1} \frac{b_k f_k}{\lambda_k} \left(\frac{e^{\lambda_k \tau} - 1}{z - e^{\lambda_k \tau}} \right) + \sum_{k=0}^{K-1} \frac{b_k f_k}{\lambda_k} \left(\frac{e^{\lambda_k \tau} - 1}{z - e^{\lambda_k \tau}} \right) - H_\tau(z)$$

¿From Step 2, if $|z| \geq 1$, then

$$|1 - H_\tau(z)| \geq \left| 1 - \sum_{k=0}^{K-1} \frac{b_k f_k}{\lambda_k} \left(\frac{e^{\lambda_k \tau} - 1}{z - e^{\lambda_k \tau}} \right) \right| - \frac{\epsilon^* - \epsilon}{4} \, .$$

Since $|z-1| \geq \gamma$, it is obvious that we can find τ^* sufficiently small so that if $\tau \in (0, \tau^*)$, then

$$\left| 1 - \sum_{k=1}^{K_3-1} b_k f_k \frac{e^{\lambda_k \tau} - 1}{\lambda_k (z - e^{\lambda_k \tau})} \right| \geq \epsilon^* \, . \tag{2.15}$$

This is because the denominators of the terms in this finite sum are all bounded away from 0 whilst the corresponding numerators tend to 0 as τ tends to 0.

Step 5 To summarize, piecing together Steps 1 - 4, we have proved that for all $\epsilon \in (0, \epsilon^*)$, there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$, (2.5) holds, as claimed. □

Remark 2.2 Closed-loop continuous time H_∞-style performance is related to the H_∞-norm of the sensitivity function $(I - G(s))^{-1}$, while closed-loop sampled-data performance is related to the H_∞-norm of the sensitivity function $(I - H_\tau(s))^{-1}$ of the related discrete time system. Theorem 2.1 then shows that the continuous time performance can be "recovered" in the sampled-data system by sampling fast enough.

We next show that if we include some mild conditions on (A, b, f), then we can conclude that the sampled data system is exponentially stable.

Corollary 2.3 *Suppose that $\kappa > 1$, i.e. the open-loop system has an unstable part, and*

1. *for all $k = 1, \ldots \kappa - 1$, $b_k \neq 0$ and $f_k \neq 0$;*
2. *if $k \neq j$, then $\lambda_k \neq \lambda_j$;*
3. *(1.6) and (1.7) hold; and*
4. *there exists $\epsilon^* > 0$ such that (2.4) holds.*

Then there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$, the closed-loop sampled data system (1.1), (1.3) is exponentially stable, in the sense that there exists $N \geq 1$ and $\nu > 0$ such that the solution of (1.1), (1.3) with $x(0) = x_0$ satisfies

$$\|x(t)\| \leq N e^{-\nu t} \|x_0\|.$$

Proof: First note that hypotheses 1. and 2. guarantee that the unstable part of (A, B, F) is controllable and observable. By Theorem 2.1, hypotheses 3. and 4. guarantee (2.5) for all small enough $\tau > 0$. Hence, from the proof of lemma 4.7 in [3], these hypotheses are sufficient to conclude that

$$\Delta_\tau := T(\tau) + \int_0^\tau T(s)BF\,ds$$

is power stable for all small enough $\tau > 0$. Then from lemma 2.3 of [3] we see that (1.1), (1.3) is exponentially stable. □

Remark 2.4 It is very easy to construct examples which satisfy (1.6) but do not satisfy either of the sets of hypotheses in [3]. For instance, if $T(t)$ is a differentiable semigroup which is not analytic, then $\{|\mathrm{Re}(\lambda_k)|\}$ is an unbounded sequence, so there exist $b \notin \ell^2$ for which (1.6) is satisfied for any $f \in \ell^2(\mathbb{C})$. In fact, even if there exists $\alpha, \beta > 0$ such that

$$\alpha \leq |\mathrm{Re}(\lambda_k)| \leq \beta$$

(so $T(t)$ is not analytic), for a given $f \in \ell^2$ there might exist $b \notin \ell^2$ (so B is not bounded) for which (1.6) is satisfied. As a trivial example, if $f_k = 1/k$, we can take $b_k = 1/k^{1/4}$, so $b \notin \ell^2$ but (1.6) is satisfied.

Remark 2.5 Suppose that the control space is $\mathbb{C}^m$, $m \in \mathbb{N}$, $m > 1$, rather than $\mathbb{C}$. Then it is not hard to modify our results. In this case $b_k \in \mathcal{B}(\mathbb{C}^m, \mathbb{C})$, i.e. b_k is a row vector of length m. Similarly, f_k is now a column vector of length m. The transfer function becomes

$$G(s) = \sum_{k=1}^{\infty} \frac{f_k b_k}{s - \lambda_k},$$

where the matrix $f_k b_k$ has norm $\|f_k\|_{\mathbb{C}^m} \|b_k\|_{\mathbb{C}^m}$. Theorem 2.1 is now true with (1.6) replaced by

$$\sum_k \frac{\|b_k\|_{\mathbb{C}^m} \|f_k\|_{\mathbb{C}^m}}{1 + |\mathrm{Re}\,\lambda_k|} < \infty\,;$$

The proof is the same, with $b_k f_k$ replaced by $\|f_k\|_{\mathbb{C}^m} \|b_k\|_{\mathbb{C}^m}$. Similarly, Corollary 2.3 is true, with the conditions that $b_k \neq 0$ and $f_k \neq 0$ for $k = 1, \ldots \kappa - 1$ replaced by conditions which guarantee that the unstable part of (A, B, F) is controllable and observable.

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