International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

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Online publication date: 13 December 2010

To cite this Article Morris, Kirsten and Rebarber, Richard(2010) 'Invariant zeros of SISO infinite-dimensional systems', International Journal of Control, 83: 12, 2573 — 2579
To link to this Article: DOI: 10.1080/00207179.2010.534177
URL: http://dx.doi.org/10.1080/00207179.2010.534177

International Journal of Control
Vol. 83, No. 12, December 2010, 2573–2579

Invariant zeros of SISO infinite-dimensional systems

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(*Received 26 June 2008; final version received 19 October 2010*)

The zeros of a finite-dimensional system can be characterised in terms of the eigenvalues of an operator on the largest closed feedback-invariant subspace. This characterisation is also valid for infinite-dimensional systems, provided that a largest closed feedback-invariant subspace exists. We generalise this characterisation of the zeros to the case when the largest closed feedback-invariant subspace does not exist. We give an example which shows that the choice of domain of the operator on this invariant subspace is crucial to this characterisation.

Keywords: infinite-dimensional systems; zeros; feedback-invariance

1. Introduction

The importance of the poles of a transfer function to system dynamics are well known. The zeros of the transfer function are also important to controller design e.g. Doyle, Francis, and Tannenbaum (1992) and Morris (2001). For example, the poles of a system controlled with a constant feedback gain move to the zeros of the open-loop system as the gain increases. Furthermore, regulation is only possible if the zeros of the system do not coincide with the poles of the signal to be tracked. Another example is sensitivity reduction – arbitrary reduction of sensitivity is only possible if all zeros lie in the open left-half-plane.

There are a number of ways to define the zeros of a system; for systems with a finite-dimensional state space all these definitions are equivalent. However, systems with delays or partial differential equation models have state space representations with an infinite-dimensional state space. Since the zeros are often not accurately calculated by numerical approximations (Lindner, Reichard, and Tarkenton 1993; Clark 1997; Cheng and Morris 2003; Grad and Morris 2003) it is useful to obtain an understanding of their behaviour in the original infinite-dimensional context. Extensions from the finite-dimensional situation are complicated not only by the infinite-dimensional state space but also by the unboundedness of the generator A. There are results on the zero behaviour of infinite-dimensional systems; see in particular the book Zwart (1989) and the recent papers Jacob, Morris, and Trunk (2007) and Morris and Rebarber (2007). However, a complete picture of the zeros of linear infinite-dimensional systems and how the several definitions of zeros relate has not been obtained (Zwart 1999).

Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on X, and let b and c be elements of X. We consider the following system on X:

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0, \tag{1}$$

with the observation

$$y(t) = Cx(t) := \langle x(t), c \rangle. \tag{2}$$

We sometimes refer to this system as (A, b, c) and for an operator K will write $A + bK$ to indicate $A + BK$ where B is the operator defined by $Bu = bu$. The transfer function for this system is $G(s) = \langle R(s, A)b, c \rangle$, where $R(s, A) := (sI - A)^{-1}$.

Definition 1.1: The invariant zeros of (1), (2) are the set of all λ such that

$$\begin{bmatrix} \lambda I - A & b \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

has a solution for some scalar u and non-zero $x \in D(A)$.

Definition 1.2: The transmission zeros of (1.1), (1.2) are the zeros of the transfer function.

As for finite-dimensional systems, every transmission zero is an invariant zero. Also if an invariant zero $z \in \rho(A)$, then z is a transmission zero (Pohjolainen 1981).

Invariant subspaces are one of the most fundamental ways to characterise the zeros of a system. The following definitions of invariance are standard (Zwart 1989), except that we do not restrict Z to being closed, for reasons that will become apparent.

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ISSN 0020–7179 print/ISSN 1366–5820 online

DOI: 10.1080/00207179.2010.534177
http://www.informaworld.com

Definition 1.3: A subspace Z of X is A-invariant if $A(Z \cap D(A)) \subset Z$.

Definition 1.4: A subspace Z of X is (A, b) feedback-invariant if there exists an A-bounded feedback K such that Z is $A+bK$-invariant.

When the choice of operators A and b is clear we will sometimes refer to an A-invariant subspace simply as invariant and to (A, b)-feedback-invariance as feedback-invariance.

Suppose that for a given system (A, b, c) a largest feedback-invariant subspace Z in the kernel of $C=\langle c, \cdot\rangle$ exists, that is, all feedback-invariant subspaces in the kernel of C are a subset of Z. Let K be a feedback so that Z is $A+bK$-invariant. Then the invariant zeros are identical to the eigenvalues of the operator $A+bK$ on Z, see Theorem 2.3.

For systems defined on a finite-dimensional space X, a largest feedback-invariant subspace in the kernel of C always exists. The situation when X is infinite dimensional is considerably more complex and was first considered in the 1980s, see in particular Curtain (1986) and Zwart (1989). In those papers the feedback K is generally assumed to be bounded, although this assumption is generalised in parts of Zwart (1989) to the assumption that $A+bK$ generates a C_0-semigroup. One complication in the analysis of infinite-dimensional systems is that even if b and C are bounded operators, an unbounded operator K may be required to obtain a largest feedback-invariant subspace (Morris and Rebarber 2007). If K is unbounded then $A+bK$ might not generate a C_0-semigroup. In Morris and Rebarber (2007) sufficient conditions for existence of a largest invariant subspace were established. No assumptions on generation of a closed-loop semigroup are needed. Conditions are also provided in Morris and Rebarber (2007) under which the trajectories of (1) with $u(t)=Kx(t)$ are well defined. The behaviour of these trajectories is often called the *zero dynamics*, and is determined by the spectrum of $A+bK$.

The existence of a largest feedback-invariant subspace is closely connected to the relative degree of the system and also depends on the regularity of c, see Theorem 2.10 in Morris and Rebarber (2007). In Section 2 we describe the relationship between the relative degree of a system (see Definition 2.4), the largest invariant subspace, and the invariant zeros. Even allowing for unbounded feedback K, a largest invariant subspace in the kernel of C might not exist. This is examined in detail in Section 3 of this article. We generalise the earlier feedback to arbitrary systems with finite relative degree and give a feedback-invariance characterisation of the zeros. Our results are illustrated with an example.

2. Zeros – largest invariant subspace

The following lemma is a re-formulation of Zwart (1989, Lemma II.25) for the case we are considering here of a single input.

Lemma 2.1: *If $Z_1 \subset D(A)$ is a linear subspace that is closed with respect to the graph norm of A and Z_2 is a linear subspace with*

$$AZ_1 \subset Z_2 + \operatorname{span}\{b\}$$

and $b \notin Z_2$ then there exists a unique A-bounded feedback K such that $(A+bK)Z_1 \subset Z_2$.

Letting Z_2 be a closed subspace of X, and $Z_1 = Z_2 \cap D(A)$, the following result follows immediately.

Theorem 2.2 (Zwart 1989, Theorem II.26)**:** *A closed subspace Z is feedback-invariant if and only if it is (A, b)-invariant, that is,*

$$A(Z \cap D(A)) \subseteq Z \oplus \operatorname{span}\{b\}.$$

The following theorem and its proof are similar to the corresponding result for finite-dimensional systems.

Theorem 2.3: *Consider a system* (1), (2). *Assume that a largest feedback-invariant subspace $Z \subset c^{\perp}$ exists and that $b \notin Z$. Let K be an A-bounded operator such that Z is $A+bK$-invariant. The set of eigenvalues of $(A+bK)|_Z$ is identical to the set of invariant zeros of the system.*

Proof: Let K be an operator such that Z is $A+bK$-invariant. For any other such operator, say $\tilde{K}$, $b(K-\tilde{K})x \in Z$ for all $x \in Z$. Since $b \notin Z$, this implies that $K|_Z$ is unique.

Suppose that λ is an eigenvalue of $(A+bK)|_Z$ with eigenvector $x_o \in D(A) \cap Z$. Since

$$\begin{bmatrix} \lambda I - A & b \\ C & 0 \end{bmatrix} \begin{bmatrix} x_o \\ -Kx_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we see that λ is also an invariant zero.

Now suppose that λ is an invariant zero of (A, b, c). Then there exists $u_o \in \mathbb{C}$ and non-zero $x_o \in D(A)$ such that

$$\begin{bmatrix} \lambda I - A & b \\ C & 0 \end{bmatrix} \begin{bmatrix} x_o \\ u_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4}$$

Clearly, $x_o \in c^{\perp}$. Also,

$$Ax_o = \lambda x_o + bu_o,$$

and so x_o is contained in an (A, b)-invariant subspace of $c^{\perp}$. By Theorem 2.2, Z is the largest such subspace, and so $x_o \in Z$. Thus, we can rewrite the first equation in (4) as

$$(\lambda I - A - bK)x_o + b(Kx_o + u_o) = 0.$$

Since $b \notin Z$ and $(\lambda I - A - bK)x_0 \in Z$, $Kx_o + u_o = 0$ and λ is an eigenvalue of $(A + bK)|_Z$. □

A difficulty which occurs in infinite-dimensional systems but does not occur in finite-dimensional systems is that the largest feedback-invariant subspace might not exist. Existence of such a subspace hinges upon the *relative degree* of (A, b, c).

Definition 2.4: (A, b, c) is of relative degree n for some positive integer n if

(1) $\limsup_{s\to\infty,\, s\in\mathbb{R}} |s^n G(s)| \neq 0$ and
(2) $\lim_{s\to\infty,\, s\in\mathbb{R}} s^{n-1} G(s) = 0$

It was shown in Morris and Rebarber (2007) that if (A, b, c) has relative degree $n+1$ and $c \in D(A^{*n})$ then the largest invariant subspace in $c^\perp$ exists. This result is a generalisation of the well-known feedback-invariance result for finite-dimensional systems (Wonham 1985).

Theorem 2.5 (Morris and Rebarber 2007, Theorem 2.10)**:** *Suppose that (A, b, c) is of relative degree $n+1$, where n is a non-negative integer, and that $c \in D(A^{*n})$. Define, for $n = 0, 1, 2, \ldots$,*

$$Z_n = c^\perp \cap (A^*c)^\perp \cap \cdots (A^{*n}c)^\perp,$$

and define $Z_{-1} = X$. Then Z_n is the largest feedback-invariant subspace in $c^\perp$. A suitable feedback is

$$Kx = -\frac{\langle Ax, A^{n*}c\rangle}{\langle b, A^{n*}c\rangle}.$$

We show in Morris and Rebarber (2007, Proposition 3.3) that the additional assumption of *uniform relative degree* (Morris and Rebarber 2007, Definition 1.5) is sufficient to ensure that the closed-loop system $\dot{x}(t) = Ax(t) + bKx(t)$, with initial data in $D(A)$, has a generalised solution which satisfies the semigroup property. Furthermore, $A + bK$ generates an integrated semigroup. There is no guarantee that the closed-loop operator $A + bK$ generates a strongly continuous semigroup, but it often does. We also show in Morris and Rebarber (2007) that if $A + bK$ does generate a C_0-semigroup on X, then it generates a C_0-semigroup on the largest feedback-invariant subspace of $c^\perp$.

This result is enough to characterise the zeros of these systems in terms of the spectrum of $A + bK$ on the largest invariant subspace.

Theorem 2.6: *Suppose that (A, b, c) is of relative degree $n+1$, where n is a non-negative integer, and that $c \in D(A^{*n})$. Then the invariant zeros of the system are the eigenvalues of $(A + bK)$ on Z_n where K and Z_n are defined in Theorem 2.5.*

Proof: The assumptions that the system has relative degree $n+1$ and $c \in D(A^{*n})$ implies that $b \notin Z_n$ (Morris and Rebarber 2007, Lemma 2.9). The result then follows from Theorems 2.3 and 2.5. □

3. Zeros – general case

Theorem 2.6 characterises the zeros for relative degree $n+1$ systems that satisfy $c \in D(A^{*n})$. For a relative degree 1 system, that is $\langle b, c\rangle \neq 0$, this assumption is always satisfied, and the largest invariant subspace is guaranteed to be $c^\perp$. For systems of relative degree $n \geq 2$, there might not be a largest feedback-invariant subspace if $c \notin D(A^{*n})$. In Morris and Rebarber (2007) an example is given of a system with $\langle b, c\rangle = 0$ and $c \notin D(A^*)$ that does not have a largest feedback-invariant subspace as defined in Definition 1.4.

Consider a relative degree 2 system. The fact that the relative degree is greater than one implies that $b \in c^\perp$. It follows from Theorem 2.2 that any element $x \in D(A)$ of an (A, b)-invariant subspace of $c^\perp$ is contained in the set

$$Z = \{z \in c^\perp \cap D(A) \mid \langle Az, c\rangle = 0\}. \tag{5}$$

The closure of Z is a natural candidate for the largest feedback-invariant subspace of $c^\perp$. If $c \in D(A^*)$, the closure of Z in X is $Z_1 = c^\perp \cap (A^*c)^\perp$, which is indeed the largest feedback-invariant subspace in $c^\perp$ if $\langle b, A^*c\rangle \neq 0$ (Theorem 2.5). In fact, Z_1 is $A + bK$-invariant, where

$$A + bK = A + \alpha b\langle Ax, A^*c\rangle, \quad \alpha = \frac{-1}{\langle b, A^*c\rangle},$$
$$D(A + bK) = \{z \in c^\perp \cap D(A) \mid \langle Az, c\rangle = 0\}.$$

Under weak conditions, this operator leads to an integrated semigroup, and the solutions remain in Z_1 when initial conditions are in $D(A_K) \subset Z_1$. In many cases, this operator generates a C_0-semigroup on Z_1 (Morris and Rebarber 2007). However, if $c \notin D(A^*)$ the situation is quite different.

Theorem 3.1 (Morris and Rebarber 2007, Theorem 4.2)**:** *If $c \notin D(A^*)$, the set Z is dense in $c^\perp$. Furthermore, $Z \neq c^\perp \cap D(A)$.*

It is tempting to hope, that even if $c \notin D(A^*)$, the operator (with some value of α)

$$A + bK = A + \alpha b\langle A^2x, c\rangle,$$
$$D(A + bK) = \{z \in c^\perp \cap D(A^2) \mid \langle Az, c\rangle = 0\}$$

is a generator, or has an extension which is a generator. However, we see from the next result that this operator is not closable, so that no extension of it is a generator of a C_0-semigroup, or even an integrated semigroup, see Neubrander (1988, Theorem 4.5).

Theorem 3.2 (Morris and Rebarber 2007, Theorem 4.4)**:** *Suppose that* $b \in X$ *and* $c \notin D(A^*)$. *Then the operator*

$$A_F x := Ax + b\langle A^2 x, c\rangle,$$
$$D(A_F) = \{x \in c^{\perp} \cap D(A^2) \mid \langle Ax, c\rangle = 0\}$$

is not closable.

We now show that even if the system has relative degree $n+1$ and $c \notin D(A^{*n})$, we can find a feedback K and a subspace Z of $c^{\perp}$ that is $(A+bK)$-invariant such that the spectrum of $A+bK$ yields the invariant zeros. This result is only true if the domain of $A+bK$ is chosen correctly. In general, such a $A+bK$ is not closable on the original Hilbert space, hence does not generate a C_0-semigroup in the original norm.

We extend the operator $\langle A\cdot, c\rangle$ and related operators to a larger set than $D(A)$ as follows. Define

$$C_A x = \lim_{s\to\infty, s\in\mathbb{R}} \langle sAR(s, A)x, c\rangle \tag{6}$$

with domain

$$D(C_A) = \{x \in X | \lim_{s\to\infty, s\in\mathbb{R}} \langle sAR(s, A)x, c\rangle \text{ exists}\}.$$

(This is the same as the extension of CA given by Weiss (1994, Definition 5.6).) It is straightforward to verify that $D(C_A) \supseteq D(A)$. If $x \in D(A)$, then $C_A(x) = \langle Ax, c\rangle$. Also, if $c \in D(A^*)$, then $D(C_A) = X$ and $C_A x = \langle x, A^* c\rangle$. Similarly, define for any integer $i > 1$,

$$C_{Ai} x = \lim_{s\to\infty, s\in\mathbb{R}} \langle s^i AR(s, A)x, c\rangle$$

with domain

$$D(C_{Ai}) = \{x \in X | \lim_{s\to\infty, s\in\mathbb{R}} \langle s^i AR(s, A)x, c\rangle \text{ exists}\}.$$

The following properties of C_{Ai} will be required.

Lemma 3.3: *Consider* $x \in D(A)$ *such that* $C_A x = 0$. *Then for all integers* $i > 1$, $x \in D(C_{Ai})$ *if and only if* $Ax \in D(C_{A(i-1)})$. *If this is the case, then for* $i > 1$

$$C_{Ai} x = C_{A(i-1)}(Ax).$$

Proof: Consider any $x \in D(A)$ and integer $i > 1$.

$$\begin{aligned}\langle s^i AR(s, A)x, c\rangle &= \langle s^{i-1}(sI - A)AR(s, A)x, c\rangle \\ &\quad + \langle s^{i-1}A^2 R(s, A)x, c\rangle \\ &= \langle s^{i-1}Ax, c\rangle + \langle s^{i-1}AR(s, A)Ax, c\rangle \\ &= s^{i-1}C_A x + \langle s^{i-1}AR(s, A)Ax, c\rangle \\ &= \langle s^{i-1}AR(s, A)Ax, c\rangle,\end{aligned}$$

since $C_A x = 0$. The result now follows from definition of C_{Ai}. □

Proposition 3.4: *Assume that* (A, b, c) *has relative degree* $n+1$ *where* $n \in \mathbb{N}$. *Then* $b \in D(C_{Ai})$ *and* $C_{Ai} b = 0$ *for all integers* $i < n$. *Also,* $\lim_{s\to\infty, s\in\mathbb{R}} s^{n+1} G(s)$ *exists for real s if and only if* $b \in D(C_{An})$. *If so,*

$$\lim_{s\to\infty} s^{n+1} G(s) = C_{An} b. \tag{7}$$

Proof: First note that since the relative degree of the system is at least 2, $\lim_{s\to\infty} sG(s) = 0$. Since

$$\lim_{s\to\infty} sG(s) = \lim_{s\to\infty} \langle sR(s, A)b, c\rangle = \langle b, c\rangle,$$

we see that $\langle b, c\rangle = 0$.

Now, for any $i \in \mathbb{N}$ such that $1 \le i \le n$,

$$\begin{aligned}s^{i+1}G(s) &= \langle s^i(sI - A)R(s, A)b, c\rangle + \langle s^i AR(s, A)b, c\rangle \\ &= s^i\langle b, c\rangle + \langle s^i AR(s, A)b, c\rangle \\ &= \langle s^i AR(s, A)b, c\rangle.\end{aligned}$$

The result follows from the definition of C_{Ai}. □

One consequence of Proposition 3.4 is that if a system has relative degree $n+1$ and if $\lim_{s\to\infty} s^{n+1} G(s)$ exists, then $C_{An} b$ is not only well defined but also non-zero.

Using the operators C_{Ai}, the space Z_n and the feedback operator K defined in Theorem 2.5 can be generalised to the case where c may not be in $D(A^{*n})$.

Theorem 3.5: *Assume that* (A, b, c) *has relative degree* $n+1$ *where* $n \ge 1$ *and* $\lim_{s\to\infty} s^{n+1} G(s)$ *exists. Let*

$$A_K x = Ax + bKx, \tag{8}$$

where

$$Kx = -\frac{C_{An}(Ax)}{C_{An}(b)} \tag{9}$$

with domain

$$D(A_K) = \{x \in D(A) \cap c^{\perp} | Ax \in D(C_{An})\}. \tag{10}$$

The space

$$Z_A = \{x \in c^{\perp} \cap D(C_{An}) \mid C_{An} x = C_{A(n-1)} x = \cdots = C_A x = 0\}$$

is A_K*-invariant.*

Proof: For $x \in D(A_K) \cap Z_A$,

$$\langle A_K x, c\rangle = \langle Ax, c\rangle - \frac{C_{An}(Ax)\langle b, c\rangle}{C_{An}(b)}.$$

Since the relative degree is at least 2, $\langle b, c\rangle = 0$. Thus,

$$\begin{aligned}\langle A_K x, c\rangle &= \langle Ax, c\rangle\\ &= 0,\end{aligned}$$

since $x \in Z_A$. Also,

$$\begin{aligned}C_{An}(A_K x) &= C_{An}(Ax) - \frac{C_{An}(Ax)C_{An}(b)}{C_{An}(b)}\\ &= C_{An}(Ax) - C_{An}(Ax)\\ &= 0.\end{aligned}$$

Since $C_{Ai}(b) = 0$ for $i < n$, we have that for any integer $i < n$,

$$\begin{aligned}C_{Ai}(A_K x) &= C_{Ai}(Ax) - \frac{C_{An}(Ax)C_{Ai}(b)}{C_{An}(b)}\\ &= C_{Ai}(Ax)\\ &= C_{A(i+1)}(x)\end{aligned}$$

from Lemma 3.3, since $x \in D(A)$ and $C_A(x) = 0$. This implies that $A_K x \in D(C_{Ai})$. Since $C_{Ai}(x) = 0$, for $i = 1, \dots, n$, $C_{Ai}(A_K x) = 0$ for all $i = 1, \dots, n$. Thus, Z_A is invariant under the operator A_K. □

If $c \in D(A^{*n})$, then K is the same A-bounded operator defined in Section 2 and $Z_A = Z_n$. In this situation Z_A is closed and is the largest A-invariant subspace in the kernel of C. Furthermore, $D(A_K) \cap Z_A$ is dense in Z_A and in many situations, A_K generates a C_0-semigroup on Z_A.

However, if the assumption $c \in D(A^{*n})$ does not hold, then Z_A is not in general closed (see Theorem 3.1 for the relative degree 2 case). Furthermore, the feedback K is not A-bounded. By Lemma 2.1, if $D(A_K)$ is contained in a subspace V closed in the graph metric on $D(A)$ then there exists a unique feedback so that V is (A, b) feedback-invariant. Moreover, this feedback is A-bounded. However, since the feedback constructed above on $D(A_K)$ is not A-bounded, it is not possible to contain $D(A_K)$ in a closed feedback-invariant subspace. Theorem 3.2 shows that for relative degree 2 systems, the operator A_K defined in (8)–(10) is not even closable.

Despite the fact that K is A-unbounded when the assumption on $c \in D(A^{*n})$ does not hold, K and A_K are the appropriate operators for defining invariant zeros in terms of the eigenvalues of a closed-loop operator.

Theorem 3.6: *Assume that the system (A, b, c) has relative degree $n+1$, $n \geq 1$ and that $\lim_{s\to\infty} s^{n+1} G(s)$ exists. The invariant zeros of (A, b, c) are the eigenvalues of A_K on $c^\perp$, where A_K is as defined in* (8), (9).

Proof: First assume that λ is an eigenvalue of A_K with eigenvector v. Note that $v \in D(A) \cap c^\perp$, so set $x = v$ and $u = -Kv$ in (3) to obtain that λ is an invariant zero of the original system.

Now assume that λ is an invariant zero. That is, there exists $u \in \mathbb{R}$ and $v \neq 0$ such that $v \in c^\perp \cap D(A)$ and

$$\lambda v - Av + bu = 0.$$

We need to first show that $v \in D(A_K)$. First

$$\begin{aligned}C_A(v) &= \langle Av, c\rangle\\ &= \lambda\langle v, c\rangle + u\langle b, c\rangle\\ &= 0,\end{aligned}$$

since the relative degree is at least 2 and $v \in c^\perp$. Thus, $v \in D(C_A)$ and since $Av = \lambda v - bu$, $Av \in D(C_A)$. Since $v \in D(A)$ and $C_A(v) = 0$, Lemma 3.3 implies that

$$\begin{aligned}C_{A2}(v) &= C_A(Av)\\ &= \lambda C_A v + u C_A b\\ &= u C_A b.\end{aligned}$$

Thus, $v \in D(C_{A2})$. Using induction, $v \in D(C_{Ai})$ and $C_{Ai}(v) = 0$, for $i = 1 \dots n$. Thus, $z \in Z_A$.

Since $\lim_{s\to\infty} s^{n+1} G(s)$ exists, $b \in D(C_{An})$ and so

$$\begin{aligned}C_{An}(Av) &= \lambda C_{An}(v) + u C_{An}(b)\\ &= u C_{An}(b).\end{aligned}$$

Thus, $v \in D(A_K)$ and we can write

$$\begin{bmatrix}\lambda I - A_K & b\\ C & 0\end{bmatrix}\begin{bmatrix}v\\ Kv + u\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

Since $b \notin Z_A$, $Kv + u = 0$ and λ is an eigenvalue of A_K on $c^\perp$ with the given domain. □

A more obvious choice of domain for the operators K and A_K is $D(A^{n+1})$ where $n+1$ is the relative degree. It is shown in the next example that, in general, it is not possible to restrict $D(A_K)$ to $D(A^{n+1})$ and obtain the invariant zeros.

Example: The following example of a controlled delay equation first appeared in Pandolfi (1986, Example 5.1).

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) - x_2(t-1)\\ \dot{x}_2(t) &= u(t)\\ y(t) &= x_1(t).\end{aligned} \tag{11}$$

The system of Equations (11) can be written in a standard state space form (1), (2), see Curtain and Zwart (1995). Choose the state space

$$X = R \times R \times L_2(-1, 0) \times L_2(-1, 0).$$

A state space realisation on X is

$$b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$D(A) := \{[r_1, r_2, \phi_1, \phi_2]^T \mid \phi_1(0) = r_1,\ \phi_2(0) = r_2,\ \phi_1 \in H^1(-1,0), \phi_2 \in H^1(-1,0)\},$$

and for $[r_1, r_2, \phi_1, \phi_2]^T \in D(A)$,

$$A(r_1, r_2, \phi_1, \phi_2) = \begin{pmatrix} \phi_2(0) - \phi_2(-1) \\ 0 \\ \dot\phi_1 \\ \dot\phi_2 \end{pmatrix}.$$

In this example $\langle b, c\rangle = 0$ and $c \notin D(A^*)$. The transfer function for this system is

$$G(s) = \frac{1 - e^{-s}}{s^2}, \tag{12}$$

so

$$\lim_{s\to\infty} s^2 G(s) = 1. \tag{13}$$

Hence the system has relative degree 2. The existence of the limit in (13) also implies that $b \in D(C_A)$ with $C_A b = 1$.

The invariant zeros of this control system can be easily verified to be $i2n\pi$, where n is any integer. We now verify that these are the eigenvalues of A_K on $c^\perp$.

We first calculate C_A. For any $x = (r_1, r_2, \phi_1, \phi_2) \in X$ define

$$\psi_2(t) = e^{st}\int_t^0 e^{-s\tau}\phi_2(\tau)\mathrm{d}\tau + \frac{1}{s} r_2 e^{st},$$

$$p_1 = \frac{1}{s}\left[r_1 + \frac{1}{s} r_2 - \psi_2(-1)\right],$$

$$\psi_1(t) = e^{st}\int_t^0 e^{-s\tau}\phi_1(\tau)\mathrm{d}\tau + p_1 e^{st}.$$

The resolvent of A is

$$R(s; A)x = \begin{pmatrix} p_1 \\ \frac{1}{s} r_2 \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix}.$$

It follows that

$$\begin{aligned}\langle AR(s; A), c\rangle &= \psi_2(0) - \psi_2(-1) \\ &= \frac{1}{s} r_2 - e^{-s}\int_{-1}^0 e^{-s\tau}\phi_2(\tau)\mathrm{d}\tau - \frac{1}{s} r_2 e^{-s}.\end{aligned}$$

Then, from the definition of C_A,

$$\begin{aligned}C_A x &= \lim_{s\to\infty} s\langle AR(s; A)x, c\rangle \\ &= r_2 - \lim_{s\to\infty} se^{-s}\int_{-1}^0 e^{-s\tau}\phi_2(\tau)\mathrm{d}\tau,\end{aligned}$$

with

$$\begin{aligned}D(C_A) = \Big\{&[r_1, r_2, \phi_1, \phi_2]^T \in X; \\ &\lim_{s\to\infty} se^{-s}\int_{-1}^0 e^{-s\tau}\phi_2(\tau)\mathrm{d}\tau \text{ exists}\Big\} \\ &\supset \{[r_1, r_2, \phi_1, \phi_2]^T \in X; \phi_2 \in H_1(-1,0)\}.\end{aligned}$$

Denote the limiting value of

$$\lim_{s\to\infty} se^{-s}\int_{-1}^0 e^{-s\tau}\psi(\tau)\mathrm{d}\tau$$

by $E_{-1}\psi$, if this limit is well defined. (If ψ is continuous at -1, $E_{-1}\psi = \psi(-1)$.) We have $C_A b = 1$ and $A_K = A + bK$, where

$$Kx = -C_A(Ax) = E_{-1}\dot\phi_2, \tag{14}$$

with

$$\begin{aligned}D(A_K) = \{&(0, r_2, \phi_1, \phi_2);\ \phi_1(0) = 0, \\ &\phi_2(0) = \phi_2(-1) = r_2,\ \phi_1 \in H_1(-1,0), \\ &\phi_2 \in H_1(-1,0),\ E_{-1}\dot\phi_2 \text{ exists}\}.\end{aligned}$$

Expanding $A_K x = \lambda x$, $x \in D(A_K)$, we obtain

$$\begin{aligned}0 &= 0 \\ E_{-1}\dot\phi_2 &= \lambda r_2 \\ \dot\phi_1 &= \lambda\phi_1 \\ \dot\phi_2 &= \lambda\phi_2.\end{aligned}$$

The only non-trivial solutions $x \in D(A_K)$ for this system of equations are for $\lambda = i2n\pi$, with

$$x = \begin{bmatrix} 0 \\ r_2 \\ 0 \\ r_2 e^{i2n\pi t} \end{bmatrix}.$$

Thus, the eigenvalues of A_K are $i2n\pi$. These are exactly the invariant zeros of the original system.

For smooth functions, where $E_{-1}\dot\psi = \dot\psi(-1)$, the feedback (14) matches that obtained in Zwart (1990, Example 4.3) by calculation on the delay differential equation. However, not only do we now have a general and formal definition of the appropriate feedback, we have a rigorous definition of the domain of the closed-loop operator. Suppose that we restrict the domain $D(A_K)$ to the more obvious

$$D(A_K) = \{x \in D(A) \cap c^{\perp} | Ax \in D(A), \langle Ax, c\rangle = 0\}.$$

This yields that A_K is invariant on Z as defined in (3.5). For this example,

$$\begin{aligned} D(A_K) = \{(0, r_2, \phi_1, \phi_2); \phi_1(0) = 0, \\ \phi_2(0) = \phi_2(-1) = r_2, \phi_1 \in H_2(-1, 0), \\ \phi_2 \in H_2(-1, 0), \dot{\phi}_1(0) = 0, \dot{\phi}_2(0) = 0.\}. \end{aligned}$$

However, with this choice of domain, A_K has only the eigenvalue 0 with eigenvector $[0, r_2, 0, r_2]^T$.

4. Conclusions

In this article, we have examined the question of the invariant zeros of SISO infinite-dimensional systems in the context of feedback-invariance. Unlike previous research on this topic, no assumptions are made on boundedness of the feedback or the smoothness of the observation operator c. The zeros can be defined in terms of an operator on a subspace of the kernel of observation operator. Unfortunately, in general this operator is not closable and so does not generate a semigroup. This means that unlike finite-dimensional systems, and many infinite-dimensional systems, the zero dynamics are not well defined.

References

Cheng, A., and Morris, K.A. (2003), 'Accurate Approximation of Invariant Zeros for a Class of SISO Abstract Boundary Control Systems', in *42nd IEEE Conference on Decision and Control*, Vol. 2, pp. 1315–1320.

Clark, R.L. (1997), 'Accounting for Out-of-bandwidth Modes in the Assumed Modes Approach: Implications on Colocated Output Feedback Control', *ASME Journal of Dynamic Systems Measurement and Control*, 119, 390–395.

Curtain, R.F. (1986), 'Invariance Concepts in Infinite Dimensions', *SIAM Journal on Control and Optimization*, 24, 1009–1031.

Curtain, R.F., and Zwart, H. (1995), *An Introduction to Infinite-Dimensional Linear Systems Theory*, New York: Springer-Verlag.

Doyle, J.C., Francis, B.A., and Tannenbaum, A.R. (1992), *Feedback Control Theory*, New York: MacMillan.

Grad, J.R., and Morris, K.A. (2003), 'Calculation of Achievable Broadband Noise Reduction using Approximations', *Engineering applications and computational algorithms (Guelph, ON, 2003), Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algor., Suppl.* 3rd International DCSIS Conference on Engineering Applications and Computational Algorithms, pp. 438–443.

Jacob, B., Morris, K.A., and Trunk, C. (2007), 'Minimum-phase Behaviour of Damped Second-order Systems with Position Measurements', *IEEE Transactions on Automatic Control*, 52, 1654–1665.

Lindner, D.K., Reichard, K.M., and Tarkenton, L.M. (1993), 'Zeros of Modal Models of Flexible Structures', *IEEE Transactions on Automatic Control*, 38, 1384–1388.

Morris, K.A. (2001), *Introduction to Feedback Control*, Burlington, MA: Academic Press.

Morris, K.A., and Rebarber, R. (2007), 'Feedback Invariance of SISO Infinite-Dimensional Systems', *Mathematics of Control, Signals and Systems*, 19, 313–335.

Neubrander, F. (1988), 'Integrated Semigroups and their Applications to the Abstract Cauchy Problem', *Pacific Journal of Mathematics*, 135, 111–155.

Pandolfi, L. (1986), 'Disturbance Decoupling and Invariant Subspaces for Delay Systems', *Applied Mathematics and Optimization*, 14, 55–72.

Pohjolainen, S. (1981), 'Computation of Transmission Zeros for Distributed Parameter Systems', *International Journal of Control*, 33, 199–212.

Weiss, G. (1994), 'Transfer Functions of Regular Linear Systems. Part I', *Transactions of the AMS*, 342, 827–854.

Wonham, W.M. (1985), *Linear Multivariable Control: A Geometric Approach*, New York: Springer-Verlag.

Zwart, H. (1989), '*Geometric Theory for Infinite Dimensional Systems*', (Lecture Notes in Control and Information Sciences, Vol. 115), Berlin: Springer Verlag.

Zwart, H. (1990) 'On the Solution of the DDP in Infinite-Dimensional Systems', *Signal Processing, Scattering and Operator Theory and Numerical Methods, Progress Systems and Control Theory*, Vol. 5, Boston: Birkhauser, pp. 363–372.

Zwart, H. (1999), 'Where are the Zeros Located?', in *Open Problems in Mathematical Systems and Control Theory*, eds V.D. Blondel, E.D. Sontag, M. Vidyasagar, and J.C. Willems (Communications on Control Engineering Series), London: Springer, pp. 285–288.