Math 901–902 Comprehensive Exam

January 23, 2008, 2:30–6:30 pm

Do two problems from each of the three sections, for a total of *six* problems.

If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

A. Groups and Character Theory

1. Let G be a nonabelian group of order $2^3 \cdot 11$ which contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Prove that there is only one such group (up to isomorphism) and find a presentation (in terms of generators and relations) for this group.

2. Let G be a group and $Z(G)$ denote its center. For $i \geq 1$ define subgroups $Z_i(G)$ inductively as follows: $Z_0(G) = \{1\}$ and $Z_i(G) = \phi_i^{-1}(Z(G/Z_{i-1}(G)))$ for $i \geq 1$, where $\phi_i : G \to G/Z_{i-1}(G)$ is the canonical homomorphism. (Thus, $Z_1(G) = Z(G)$.) A group is called *nilpotent* if $Z_i(G) = G$ for some $i \geq 0$.

   (a) Prove that $Z_i(G)$ is a characteristic subgroup of G for all $i \geq 0$.

   (b) Let G be a nilpotent group. Prove that any subgroup of G and any quotient of G are also nilpotent.

   (c) Prove that any nilpotent group is solvable.

3. Find two finite non-isomorphic groups G_1 and G_2 which have the same (complex) character table. Justify your answer as completely as you can. (Here is a precise definition of "having the same character table": for a finite group G let $\mathrm{Irr}_{\mathbb{C}}(G)$ denote the set of irreducible complex characters of G and Conj(G) denote the set of conjugacy classes of G. To say G_1 and G_2 have the same character table, we mean there are bijective maps $\phi : \mathrm{Irr}_{\mathbb{C}}(G_1) \to \mathrm{Irr}_{\mathbb{C}}(G_2)$ and $\psi : \mathrm{Conj}(G_1) \to \mathrm{Conj}(G_2)$ such that $\phi(\chi)(\psi(c)) = \chi(c)$ for all $\chi \in \mathrm{Irr}_{\mathbb{C}}(G_1)$ and $c \in \mathrm{Conj}(G_1)$.)

B. Field and Galois Theory

4. Prove the following version of the Primitive Element Theorem: Let E/F be an algebraic field extension. Then $E = F(\alpha)$ for some $\alpha \in E$ if and only if there exist only finitely many fields between E and F.

5. Let E be the splitting field of $f(x) = x^6 + 5$ over $\mathbb{Q}$.

   (a) Prove that $[E : \mathbb{Q}] = 12$.

   (b) Find field generators (over $\mathbb{Q}$) for every subfield F of E such that $[F : \mathbb{Q}] = 3$. Central to this part is to prove that you have accounted for *every* such subfield.

6. Let $f(x) = x^4 + 6x^3 + 32x^2 + 17x - 15 \in \mathbb{Q}[x]$ and $\alpha \in \mathbb{C}$ a root of $f(x)$. Prove that $\mathbb{Q}(\alpha)$ has no subfield of degree 2 over $\mathbb{Q}$. (One approach is to consider the Galois groups of $f(x)$ over $\mathbb{Z}_2$ and $\mathbb{Z}_3$.)

C. Rings and Modules

7. Let R be a commutative ring with a unique maximal ideal. Prove that every finitely generated projective R-module is free.

8. Let G be a finite group and F a field such that $\operatorname{char} F$ does not divide $|G|$.

   (a) Prove that if G is abelian then $F[G]$ has no nonzero nilpotent elements.

   (b) Suppose F is algebraically closed and G is nonabelian. Prove that $F[G]$ has infinitely many nilpotent elements.

9. Let R be a commutative (not necessarily Noetherian) ring and M and N finitely generated R-modules. Suppose M has finite length (i.e., has a composition series). Prove that $M \otimes_R N$ has finite length.

10. Let R be a ring and E a left R-module. Let $R' = \operatorname{End}_R(E)$ and $R'' = \operatorname{End}_{R'}(E)$. Let $\lambda : R \to R''$ be the natural ring homomorphism given by $\lambda(r) = \ell_r$ where ℓ_r is left multiplication by r on the module E. Prove Reiffel's theorem: If R is simple (i.e., no nontrivial two-sided ideals) and E is a non-zero left ideal of R then λ is an isomorphism.