Ph.D. Comprehensive Exam
Algebra: 901-902. June 7, 1996.

Do a total of six problems: three from Sections I and II combined, and three from sections III and IV combined. Do no more than two problems from any one section. In sections III and IV, all rings are assumed to be commutative with identity.

Section I: Groups

1. Let G be a group of order $5 \cdot 7 \cdot 19^2$.

(a) Prove that if the sylow 19-subgroup is cyclic then G is cyclic.

(b) Prove there exists a non-abelian group of order $5 \cdot 7 \cdot 19^2$.

2. Let $p \leq q \leq r$ be primes. Prove that any group of order pqr is solvable.

Section II: Fields

3. Prove that any finite separable field extension has a primitive element.

4. Let ω be a primitive 15th root of unity.

(a) Find the minimal polynomial of ω over $\mathbf{Q}$.

(b) Find the Galois group of $\mathbf{Q}(\omega)/\mathbf{Q}$.

(c) Find field generators for all the subfields of $\mathbf{Q}(\omega)$ of degree 4 over $\mathbf{Q}$.

5. Let E/K be a separable field extension of degree p, where p is a prime. Suppose $f(x) \in K[x]$ is an irreducible polynomial which has more than one root in E. Prove that $f(x)$ splits in E.

Section III: General rings and modules

6. Consider the following commutative diagram of R-modules and R-linear maps. Assume the rows are exact.

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\
 & & & & \downarrow g & & \downarrow h \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N
\end{array}$$

(a) Prove there exists a unique R-module homomorphism $f\colon A \to L$ making the diagram commute.

(b) Suppose g is surjective and h is injective. Prove that f is surjective.

7. Let P be a finitely generated projective R-module and define $\phi\colon \operatorname{Spec} R \to \mathbf{N}$ (where $\mathbf{N}$ is the set of nonnegative integers) by $f(q) = \operatorname{rank} P_q$ for all $q \in \operatorname{Spec} R$. If $\mathbf{N}$ is given the discrete topology, prove that f is a continuous map.

8. State and prove Nakayama's Lemma.

Section IV: Noetherian rings and modules

9. Let R be a reduced Noetherian ring.

(a) Prove that the set of associated primes of R (i.e., of $R/(0)$) is the same as the set of minimal primes of R (i.e., primes minimal over (0)).

(b) Prove that R_p is a field for all associated primes p of R.

10. Let R be a commutative ring which is finitely generated as an algebra over a field k. Let M be an R-module of finite length. Prove that M is a finite dimensional k-vector space.

11. Let $R \subset S$ be an integral ring extension where R is Noetherian and S is a finitely generated R-algebra. Let I be an ideal of R and $J = IS \cap R$ (which is also an ideal of R). Prove that there exists an integer k such that for all $n \geq k$, $I^n \subseteq J^n \subseteq I^{n-k}$.