Ph.D. Comprehensive Exam
Algebra: 901–902. June 1997.

Do exactly **two** problems from each section (for a total of **six** problems). If you work on more than six problems, or on more than two from any one section, clearly indicate which you want graded. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section I: Groups

1. If G is a group of order $616 = 11 \cdot 8 \cdot 7$, show that G has a normal subgroup of order 11.

2. Let A_5 denote the alternating subgroup of S_5.

(a) Show that A_5 has 4 conjugacy classes, 2 of order 12 and one each of orders 15 and 20.
(b) Use (a) to prove that A_5 is simple.

3.

(a) If F is a free group and $h : G \to F$ is a surjective homomorphism of groups, show that G is isomorphic to a semidirect product of $\ker(h)$ with F.
(b) Show that (a) can fail if F is not free.

Section II: Rings and Modules

4. Prove that a finitely generated projective module over a local ring is free.

5. If M is a Noetherian module over a commutative ring R with $1 \neq 0$, prove that a surjective homormophism $f : M \to M$ is an isomorphism.

6. Let R be a commutative Noetherian ring with $1 \neq 0$. Let P be a prime ideal of R and let M be a finitely generated R-module. If M_P denotes the localization of M at P and M_x denotes the localization of M obtained by inverting an element x of R, prove that if M_P is free, there is an element $x \in R - P$ such that M_x is free.

Section III: Fields and Galois Theory

In this section and the next $\mathbb{Q}$ and $\mathbb{C}$ denote the fields of rational and complex numbers, respectively.

7. Let F/K be a finite dimensional extension of fields. Show that $F = K(a)$ for some $a \in F$ if and only if there are only finitely many fields L with $K \subset L \subset F$.

8. Give generators for every subfield of the field $\mathbb{Q}(u)$, where u is i times the real fourth root of 5.

9. Let F be a field extension of the field E of two elements such that $[F : E] = 12$. Determine with justification the number of polynomials irreducible over E with a root in F.