Math 901-902 Comprehensive Exam

May 30, 2001   1-5pm

Do two problems from each of the three sections, for a total of *six* problems.

I Group Theory

I.1 Classify all groups of order 2001 up to isomorphism. (By the way, $2001 = 3 \cdot 23 \cdot 29$.)

I.2 Suppose a finite group G acts *transitively* on a finite set S with $|S| \geq 2$. Prove there is a $g \in G$ such that $g \cdot s \neq s$ for all $s \in S$.

I.3 Let H and K be subgroups of a finite group G. Suppose $[G : H] = p$, where p is prime, and p is strictly smaller than every prime divisor of $|K|$. Prove $K \leq H$.

II Field Theory and Galois Theory

II.1 Suppose $K \subset E_1$ and $K \subset E_2$ are finite Galois field extensions. (Assume E_1 and E_2 are each contained in some fixed algebraic closure $\overline{K}$ of K.) Prove $[E_1E_2 : K] \mid [E_1 : K] \cdot [E_2 : K]$.

II.2 Let F be the splitting field of $f(x) = x^4 + 4x^2 + 2$ over $\mathbb{Q}$. Find the lattice of subfields of F. (*Hint*: If $\pm\alpha, \pm\beta$ are the roots of $f(x)$, consider $\frac{\alpha^2-\beta^2}{\alpha\beta}$.)

II.3 Suppose $K \subset F$ is a finite Galois field extension and $p^n \mid [F : K]$, where p is prime. Prove that there exists an intermediate field extension $K \subset E \subset F$ such that $[F : E] = p^n$.

III Rings and Modules

III.1 Let R be a commutative ring (with 1) and suppose $S \subset R$ a multiplicatively closed subset. Suppose the natural ring map $R \to S^{-1}R$ defines an integral ring extension. (In particular, assume the map $R \to S^{-1}R$ is injective.) Prove that $R \to S^{-1}R$ is an isomorphism.

III.2 Let R be a commutative ring (with 1) and assume every R-module is injective.

(a) Prove that if R is an integral domain, then R must be a field.

(b) Give an example showing that the integral domain hypothesis in (a) is necessary.

III.3 Let A and B be commutative rings (with 1) and let $A \to B$ be a ring map. Suppose P is a projective A-module and prove $P \otimes_A B$ is a projective B-module.