Math 901–902 Comprehensive Exam

June 2, 2009, 2–6 pm

Instructions: Do two problems from each of the three sections, for a total of six problems. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Part I: Groups and Character Theory

(I.1) Let G be a group of order 21.

- (a) Determine all possible values of n, where n is the number of conjugacy classes of G. (I.e., for each positive integer n, determine whether or not there is a group of order 21 having exactly n conjugacy classes.)

- (b) Determine the possible decompositions of the group ring $\mathbf{C}[G]$ as a product of matrix rings. (I.e., find all possible products $\Pi_{i=1}^{r} M_{n_i}(D_i)$ which are isomorphic as rings to $\mathbf{C}[G]$ for some group G of order 21, where $\mathbf{C}$ is the field of complex numbers, each D_i is a division ring, and $M_{n_i}(D_i)$ is the ring of $n_i \times n_i$ matrices with entries in D_i.)

(I.2) For this problem you may assume that any finite group having a solvable quotient by a solvable normal subgroup is solvable. You may assume Sylow's Theorems. You may assume that if the order of a finite group G is the product of at most three (possibly non-distinct) primes, then G is solvable. You may assume that the commutator subgroup of a group is normal. You may also assume that A_n is simple for each $n > 4$. Prove everything else that you use or claim in your arguments.

- (a) If p is prime and $i \geq 0$ is an integer, prove that a group G of order p^i is solvable.

- (b) Find the least positive integer n such that there is a non-solvable group of order n. Justify your answer; in particular, justify that all groups of order less than n are solvable.

(I.3) Find all integers $0 < n < 20$ such that there exists a non-nilpotent group of order n. Justify your answer. (You may assume whatever general facts you know about nilpotent groups, but explicitly state any fact you use. If you claim a particular group is or is not nilpotent, either give a proof or cite a general fact that justifies your claim.)

Part II: Fields and Galois Theory

(II.4) Let F/k be an extension of fields such that $|k| = 3^6$ and $|F| = 3^{60}$. How many elements $\alpha \in F$ are there with $k(\alpha) = F$? Justify your answer.

(II.5) Let F be a finite (but not necessarily Galois) extension of a field K. Given a subgroup $G < \mathrm{Aut}_K(F)$, let F^G denote the intermediate field $\{f \in F \,:\, g(f) = f \text{ for all } g \in G\}$.

(a) Let $G < \mathrm{Aut}_K(F)$ be a subgroup. Prove Artin's theorem that F/F^G is a finite Galois extension with Galois group G.

(b) Let S be the set of subgroups of $\mathrm{Aut}_K(F)$ and let I be the set of intermediate fields of the extension. Define a map $\phi : S \to I$ for any $G \in S$ by setting $\phi(G) = F^G$. Show that ϕ is always injective and give an example to show that ϕ need not be surjective.

(II.6) Determine the group $\mathrm{Aut}(\mathbf{R})$ of field automorphisms of the reals. [Hint: Prove for any automorphism σ that $\sigma(x) < \sigma(y)$ for any reals $x < y$.]

Part III: Rings and Modules

(III.7) Let G be a finite group and let $\mathbf{C}$ denote the field of complex numbers. Show that $R = \mathbf{C}[G]$ has a nonzero nilpotent element if and only if G is a nonabelian group.

(III.8) Let A be a commutative ring with $1 \neq 0$. Let M and N be A-modules. Show that $M \oplus N$ is flat if and only if M and N are flat.

(III.9) Compute the number of elements in the $\mathbf{Z}$-module $A \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(M \oplus N, P \oplus Q)$, where $A = S^{-1}\mathbf{Z}$, $S = \{1, 5, 5^2, \ldots\}$, $M = \mathbf{Z}/9\mathbf{Z}$, $N = \mathbf{Z}/5\mathbf{Z}$, $P = \mathbf{Z}/3\mathbf{Z}$ and $Q = \mathbf{Z}/25\mathbf{Z}$. Justify your answer.