Real Analysis Comprehensive Examination–Math 921/922

Thursday, January 21, 2010, 2:00-6:00p.m., 347 Avery Hall

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Unless otherwise indicated $(X, \mathcal{M}, \mu)$ is a measure space.

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(1) a) Assume that μ is a finite measure. With $q \in (1, \infty)$, let $\{f_k\}_{k=1}^{\infty} \subset L^q(X, \mu)$ and $f \in L^q(X, \mu)$ be given. Suppose also that
- $\sup_{k\in\mathbb{N}} \|f_k\|_{L^q} < \infty$ and
- $f_k(x) \to f(x)$ for a.e. $x \in X$.

Prove that $f_k \to f$ in L^p for each $p \in [1, q)$.
b) Must the result in part a) remain true if μ is only assumed to be a σ-finite measure? (Justify your answer.)

(2) Assume that μ is a finite measure, and let $f : X \to [0, \infty)$ be an $\mathcal{M}$-measurable function. Suppose that $g : [0, \infty) \to [0, \infty)$ is an increasing function. Prove that

$$\int_X g(f(x))\,\mathrm{d}\mu \geq \int_0^\infty g'(t)\mu(\{x \in X \,:\, f(x) > t\})\,\mathrm{d}t.$$

(You may assume without proof that the composition $g \circ f$ is $\mathcal{M}$-measurable.)

(3) Provide all technical details to evaluate $\displaystyle\lim_{k\to\infty} \int_{[0,\infty)} \frac{\cos(x/k)}{(1 + (x/k))^k}\,\mathrm{d}x$.

(4) With m the Lebesge measure on the Lebesgue measurable sets $\mathcal{L}$ in $\mathbb{R}$, define the signed measure $\nu : \mathcal{L} \to \overline{\mathbb{R}}$ by $\nu(E) := \displaystyle\int_E x(x-1)\mathrm{e}^{-x^2}\,\mathrm{d}x$.
a) Provide a Hahn decomposition of $\mathbb{R}$ with respect to ν.
b) Provide the Jordan decomposition of ν.
c) Compute $\frac{d\nu}{dm}$.

(5) With m the Lebesgue measure on the Borel sets in $\mathbb{R}^n$, let $f \in L^1(\mathbb{R}^n, m)$ be given. Recall the definition of the Hardy-Littlewood maximal function of f:

$$Mf(x) := \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)|\,\mathrm{d}m(y).$$

a) Prove that, for each $t \geq 0$, the set $\{x \in \mathbb{R}^n \,:\, Mf(x) > t\}$ is open.
b) Argue that the function Mf is Borel measurable.

(6) Let X be a nonempty set, and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on X. Define $\nu^* : \mathcal{P}(X) \to [0, \infty]$ by

$$\nu^*(E) := \inf\{\mu^*(F) : E \subseteq F \text{ and } F \text{ is } \mu^*\text{-measurable}\}.$$

a) Prove that ν^* is an outer measure on X.
b) Prove that if E is μ^* measurable, then E is ν^*-measurable and $\mu^*(E) = \nu^*(E)$.

(7) Let m be the Lebesgue measure on the Borel sets in $\mathbb{R}$. Define $\{f_k\}_{k=1}^{\infty} \subset L^\infty(\mathbb{R}, m)$ by $f_k(x) := k\chi_{[0,k]}(x)$. Justify your answers to the following questions.
a) Does $f_k \to 0$ in measure?
b) Does $f_k(x) \to 0$ for a.e. $x \in \mathbb{R}$?
c) Does $f_k \to 0$ in L^1?
d) Does $f_k \rightharpoonup 0$ in L^1?

(8) Let $Y \subseteq X$ be given. Show that $\mathcal{N} := \{E \cap Y \,:\, E \in \mathcal{M}\}$ is a σ-algebra of subsets of Y.