Real Analysis Comprehensive Examination–Math 921/922

Friday, January 21, 2011, 2:00-6:00p.m.

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Throughout the exam, the Lebesgue measure is denoted by m, $\mathcal{B}_{\mathbb{R}}$ denotes the Borel σ-algebra on $\mathbb{R}$, and $(X, \mathcal{M}, \mu)$ denotes a general measure space.

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(1) Let $E \subset \mathbb{R}$ be a Lebesgue measurable set such that $m(E) = \infty$. Prove that:
a) $E = V \setminus N_1$, where V is a $\mathcal{G}_\delta$ set and N_1 is a null set for m.
b) $E = H \cup N_2$, where H is an $\mathcal{F}_\sigma$ set and N_2 is a null set for m.

(2) Let $\{f_n\}$ and f be $\mathcal{M}$-measurable functions on $(X, \mathcal{M}, \mu)$, $n \in \mathbb{N}$, such that $f_n \geq 0$ and $f_n \longrightarrow f$ in measure. Prove that: $\int_X f d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$. (Hint: First show that every subsequence of $\{f_n\}$ converges to f in measure).

(3) Let $f : [0, 1] \to \mathbb{R}$ be a Borel measurable function and let $g_n(x) = (\cos f(x))^{2n}$, $x \in [0, 1]$, $n \in \mathbb{N}$.
a) Carefully explain why g_n is Borel measurable for every $n \in \mathbb{N}$.
b) Show all technical details in evaluating: $\displaystyle\lim_{n\to\infty} \int_{[0,1]} g_n(x)dm$.

(4) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Assume that $f_n, f : X \to \mathbb{R}$ are $\mathcal{M}$-measurable functions such that $\|f_n\|_2$ and $\|f\|_2 \leq 1$, $n \in \mathbb{N}$; where as usual, $\|f\|_2^2 := \int_X |f|^2 d\mu$. Further assume that: $f_n \longrightarrow f$, μ-a.e.
a) Prove that: $f_n \longrightarrow f$ in $L^p(X, \mathcal{M}, \mu)$, for each $p \in [1, 2)$. (Hint: Egoroff's Theorem).
b) Provide a counterexample showing that the result in part a) fails if $p = 2$.

(5) Let $g \in L^\infty(\mathbb{R}, m)$ and $p \in [1, \infty]$ be fixed. For $f \in L^p(\mathbb{R}, m)$, define $T(f)(x) = g(x)f(x)$. Prove that $T : L^p(\mathbb{R}, m) \longrightarrow L^p(\mathbb{R}, m)$ is a bounded linear map and that $\|T\| = \|g\|_\infty$. (Don't forget to address the case $p = \infty$).

(6) Let $f \in L^1((0, 1), \mathcal{B}_{(0,1)}, m)$ and $g(x) := \displaystyle\int_{(x,1)} \frac{1}{y} f(y)dm(y)$, $x \in (0, 1)$. Prove that: $g \in L^1((0, 1), \mathcal{B}_{(0,1)}, m)$ and $\displaystyle\int_{(0,1)} g(x)dm(x) = \int_{(0,1)} f(y)dm(y)$.

(7) Let $\{r_k\}_{k=1}^\infty$ be an enumeration of the rational numbers in $\mathbb{R}$. Define $F : \mathbb{R} \longrightarrow \mathbb{R}$ by $F(x) := \sum_{\{n\in\mathbb{N} : r_n \leq x\}} 2^{-n}$.
a) Prove that F is continuous at each irrational number but discontinuous at each rational number.
b) Prove that F is everywhere right-continuous.
c) Show that F' exists a.e. and $F' = 0$ a.e.. (Hint: consider the Lebesgue-Stieltjes measure μ_F.)

(8) Let $f : [a, b] \to \mathbb{R}$ be a C^∞-function on $[a, b]$ (i.e., f is infinitely differentiable on $[a, b]$) with the property that: for every $x \in [a, b]$ there exists an integer $n_x \in \mathbb{N}$ such that $f^{(n_x)}(x) = 0$. Here, $f^{(n_x)}$ stands for the derivative of order n_x. Prove that f is a polynomial on $[a, b]$. (Suggestions: For each $n \in \mathbb{N}$, put $A_n = \{x \in [a, b] : \ f^{(n)}(x) = 0\}$ and
$E = \{x \in [a, b] : \ \text{there does not exist a neighborhood of } x \text{ on which } f \text{ is a polynomial}\}$.
Use a Theorem of Baire on E).