Real Analysis Comprehensive Examination–Math 921/922

Wednesday, January 21, 2015, 12:00-6:00p.m.

• Work 6 out of 8 problems. • Each problem is worth 20 points, parts of each problem don't necessarily carry the same weight.
• Write on one side of the paper only and hand your work in order.
• Throughout the exam, the Lebesgue measure is denoted by m, $\mathbb{B}_{\mathbb{R}}$ denotes the Borel σ-algebra on $\mathbb{R}$, and $(X, \mathcal{M}, \mu)$ denotes a general measure space.

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(1) Let $E \subset [0,1]$ be a Lebesgue measurable set with $m(E) > 0$, and let $0 < \epsilon < 1$ be given.
a) Prove that there exists an open interval I such that $m(E \cap I) > \epsilon\, m(I)$.
b) Assume $A \subset [0,1]$ is a Lebesgue measurable set such that $m(A \cap I) \geq \frac{1}{10} m(I)$ for every open interval $I \subset [0,1]$. Prove that $m(A) = 1$.

(2) Let $g : [0,1] \times [0,1] \to \mathbb{R}$ be given by: $g(x,y) = xy^3$, $(x,y) \in [0,1]^2$. Define: $\mu(E) = \mathbf{m}(g^{-1}(E))$, $E \in \mathbb{B}_{\mathbb{R}}$, where $\mathbf{m}$ denotes the two dimensional Lebesgue measure on $\mathbb{R}^2$.
a) Show that μ is a Borel measure on $\mathbb{R}$.
b) Show all details in evaluating the integral $\int_{\mathbb{R}} t^2\, d\mu(t)$.
(Hint: first show $\mu(E) = \int_{[0,1]^2} \chi_E \circ g\, d\mathbf{m}$, for all $E \in \mathbb{B}_{\mathbb{R}}$).

(3) Assume $f : [0,1] \to \mathbb{R}$ is a continuous function. Prove the limit:

$$\lim_{n\to\infty} \int_{[0,1]} f(x^n) dm$$

exists and evaluate the limit. Does the limit always exist in $\mathbb{R}$ if we only assume $f \in L^1([0,1], m)$?

(4) Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be $\mathbb{B}_{[0,1]^2}$-measurable such that the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists for each $(x,t) \in [0,1]^2$, and $M := \sup_{(x,t)\in[0,1]^2} \left|\frac{\partial f}{\partial t}(x,t)\right| < \infty$.
Prove that $\frac{\partial f}{\partial t}$ is $\mathbb{B}_{[0,1]^2}$-measurable and $\frac{d}{dt}\int_{[0,1]} f(x,t) dm(x) = \int_{[0,1]} \frac{\partial f}{\partial t}(x,t) dm(x)$..

(5) Let $f, f_n : [0,1] \to \mathbb{R}$ such that $f, f_n \in L^2([0,1], \mathbb{B}_{[0,1]}, m)$, for every $n \in \mathbb{N}$. Prove or disprove the following statements:
a) If $f_n \to 0$ a.e. $[0,1]$ then $\|f_n\|_2 \to 0$.
b) If $f_n \to 0$ in $L^2([0,1], m)$, then $f_n \to 0$ in measure.
c) If $f_n \to f$ weakly in $L^2([0,1], m)$ and $\|f_n\|_2 \to \|f\|_2$, then $\|f_n - f\|_2 \to 0$.
d) If $\|f_n - f\|_2 \to 0$, then $\|f_n - f\|_p \to 0$ for every $p \in [1,2)$.

(6) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel measurable functions such that $f, g \in L^1(\mathbb{R}, m)$.
a) Briefly explain why the mapping $(x,y) \mapsto f(x-y)g(y)$ is $\mathbb{B}_{\mathbb{R}^2}$-measurable.
b) Define: $(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y) dm(y)$. Prove that $f * g \in L^1(\mathbb{R}, m)$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1 .$$

(7) Let μ and ν be σ-finite positive finite measures on a measurable space $(X, \mathcal{M})$ with $\nu \ll \mu$. Put $\lambda = \mu + \nu$. Prove that $f = \frac{d\nu}{d\lambda}$ exists, $0 \leq f < 1$ μ-a.e., and that $\frac{d\nu}{d\mu} = \frac{f}{1-f}$ μ-a.e.

(8) Let X be a compact Hausdorff topological space with the following property:
If $E \subseteq X$ is open then its closure $\overline{E}$ is also open. Let $C(X)$ denotes the space of all continuous $\mathbb{C}$-valued functions on X. Let $f \in C(X)$ and $\epsilon > 0$ be given. Prove that there exist $\lambda_1, ..., \lambda_n \in \mathbb{C}$, $E_1, ..., E_n \subseteq X$ such that $\chi_{E_j} \in C(X)$ for every j and

$$\sup_{x\in X} \left| f(x) - \sum_{j=1}^{n} \lambda_j \chi_{E_j}(x) \right| < \epsilon.$$