Real Analysis Comprehensive Examination–Math 921/922

Tuesday, June 1, 2010, 12:30-6:30p.m., Avery Hall 119

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order. • Throughout the exam, the Lebesgue measure is denoted by m, $\mathcal{B}_{\mathbb{R}}$ denotes the Borel σ-algebra on $\mathbb{R}$, and $(X, \mathcal{M}, \mu)$ denotes a general measure space.

(1) Let X be an uncountable set. For each $E \in \mathcal{P}(X)$ define $\mu^*(E) = 0$ if E is countable and $\mu^*(E) = 1$ if E is uncountable. Let $\mathcal{A} = \{E \subset X : \ E \text{ is countable or } E^c \text{ is countable}\}$.
a) Show that $\mathcal{A}$ is σ-algebra.
b) Prove that μ^* is an outer measure on X, and determine the class of μ^*-measurable sets.

(2) Let $f : X \to \overline{\mathbb{R}}$ be such that $f \in L^1(X, \mathcal{M}, \mu)$.
a) (4 points) Prove that $\mu(A) = 0$, where $A = \{x \in X : \ |f(x)| = \infty\}$.
b) (4 points) Prove that the set $B = \{x \in X : \ f(x) \neq 0\}$ is σ-finite.
c) (12 points) Prove that: for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $E \in \mathcal{M}$ with $\mu(E) < \delta$, then $\int_E |f| d\mu < \epsilon$.

(3) Show all technical details in evaluating: $\lim_{n\to\infty} \int_{[a,\infty)} \frac{n}{1+n^2x^2}\, dm$, when $a > 0$, $a = 0$, $a < 0$. How does this accord with the various convergence theorems?

(4) Let ν be the signed measure on $\mathcal{B}_{\mathbb{R}}$ that is given by:
$\nu(E) := \int_E e^{-|x|} dm + \sum_{\{n\in\mathbb{N}:\ n\in E\}} (-1)^n \frac{3}{2^n}; \ E \in \mathcal{B}_{\mathbb{R}}$.
With justification, find
a) $\nu(0, 1]$.
b) The Hahn decomposition for ν.
c) The Jordan decomposition of ν.
d) $|\nu|$, the total variation of ν.
e) The Lebesgue decomposition of $|\nu|$ with respect to m.
f) $D\nu(x)$, the point-wise derivative of ν with respect to m.

(5) Let $f : \mathbb{R} \to [0, \infty)$ be Borel measurable and define $\phi : (0, \infty) \to [0, \infty]$ by $\phi(y) = m\{x \in \mathbb{R} : \ f(x) > y\}$. Prove that: $\int_{\mathbb{R}} f(x)^2\, dm(x) = 2 \int_{(0,\infty)} y\, \phi(y)\, dm(y)$.

(6) Let $f : [a, b] \to \mathbb{R}$ be a C^∞-function on $[a, b]$ (i.e., f is infinitely differentiable on $[a, b]$) with the property that: for every $x \in [a, b]$ there exists an integer $n_x \in \mathbb{N}$ such that $f^{(n_x)}(x) = 0$. Here, $f^{(n_x)}$ stands for the derivative of order n_x. Prove that f is a polynomial on $[a, b]$. Can you extend the result if $[a, b]$ is replaced by $\mathbb{R}$? Suggestions: For each $n \in \mathbb{N}$, put $A_n = \{x \in [a, b] : \ f^{(n)}(x) = 0\}$ and
$F = \{x \in [a, b] : \ \text{there does not exist a neighborhood of } x \text{ on which } f \text{ is a polynomial}\}$.
Use a Theorem of Baire on F. Then, invoke Taylor's Theorem.

(7) Let $f_n, f \in L^p([0, 1], \mathcal{B}_{[0,1]}, m)$; $n \in \mathbb{N}$, be $\mathbb{R}$-valued functions, where $1 < p < \infty$. Prove that $f_n \to f$ weakly in $L^p([0, 1], \mathcal{B}_{[0,1]}, m)$ if and only if $\{f_n\}$ is bounded in $L^p([0, 1], \mathcal{B}_{[0,1]}, m)$ and $\int_E f_n dm \to \int_E f dm$, for every set $E \in \mathcal{B}_{[0,1]}$.

(8) Construct, with justification, two real-valued functions F and G on $[0, 1]$ such that:
a) F is absolutely continuous and strictly increasing on $[0, 1]$; but $F' = 0$ on a set of positive measure.
b) G is absolutely continuous on $[0, 1]$; but G is not monotonic on any subinterval of $[0, 1]$.
Suggestion: There exists a Borel set $A \subset [0, 1]$ such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0, 1]$. You may use this fact without a proof.