Real Analysis Comprehensive Examination–Math 921/922

May 31, 2011, 12:00-6:00 p.m.

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.

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(1) Let h be a bounded Lebesgue measurable function on $\mathbb{R}$ having the property that $\lim_{n\to\infty} \int_E h(nx)dx = 0$, for every Lebesgue measurable subset E of finite measure. Show that for every $f \in L^1(\mathbb{R})$, we have $\lim_{n\to\infty} \int_{\mathbb{R}} f(x)h(nx)dx = 0$.

(2) Let E be a Lebesgue measurable subset of $\mathbb{R}$ and consider the function defined by $f_E(x) := |E \cap (-x, x)|$, for $x \in [0, \infty)$, where $|A|$ denotes the Lebesgue measure of A. Show that

(a) f is a uniformly continuous map taking $[0, \infty)$ onto $[0, |E|)$;

(b) $\lim_{x\to\infty} f_E(x) = |E|$.

(3) Let $(X, \mathcal{M}, \mu)$ be a finite positive measure space. Suppose $f \in L^\infty(X)$ is such that $\|f\|_\infty > 0$. For $n \in \mathbb{N}$, put $\alpha_n := \int_X |f|^n d\mu = \|f\|_n^n$. Show that

(a) $\|f\|_n \to \|f\|_\infty$, as $n \to \infty$ and

(b) $\alpha_{n+1}/\alpha_n \to \|f\|_\infty$ as $n \to \infty$.

(4) Let $\mathcal{A}$ be a $\sigma-$algebra on a set X and assume $\mu : \mathcal{A} \to [0, \infty]$ has the following properties:

(a) If $A_1, A_2 \in \mathcal{A}$ with $A_1 \cap A_2 = \emptyset$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

(b) If $\{A_n\}_{n=1}^\infty$ is a sequence in $\mathcal{A}$ such that $A_{n+1} \subset A_n$, for all $n \in \mathbb{N}$, and $\cap_{n=1}^\infty A_n = \emptyset$, then $\lim_{n\to\infty} \mu(A_n) = 0$.

Prove that μ is a positive measure on $(X, \mathcal{A})$.

(5) Let μ and ν be two regular Borel measures on $\mathbb{R}^n$. Show that $\mu \geq \nu$ (i.e. $\mu(A) \geq \nu(A)$ for every Borel set A) if and only if $\int f d\mu \geq \int f d\nu$ for each nonnegative $f \in C_c(\mathbb{R}^n)$. (You may assume that for every compact set K and open set U with $K \subset U$ there exists $f \in C_c(\mathbb{R}^n)$ such that $\chi_K \leq f \leq \chi_U$.)

(6) Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. Define

$$\mathcal{A}_\sigma := \left\{ A \in \mathcal{P}(X) : A = \bigcup_{k=1}^{\infty} A_k \text{ with } \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A} \right\},$$

and

$$\mathcal{A}_{\sigma\delta} := \left\{ A \in \mathcal{P}(X) : A = \bigcap_{k=1}^{\infty} A_k \text{ with } \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}_\sigma \right\}.$$

Let $\widetilde{\mu}$ be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

(a) Prove that for any $E \subseteq X$ and $\varepsilon > 0$, there exists an $A \in \mathcal{A}_\sigma$ such that $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.

(b) Let $E \subseteq X$ satisfying $\mu^*(E) < +\infty$ be given. Prove that E is μ^*-measurable if and only if there exists a $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq B$ and $\mu^*(B \backslash E) = 0$.

(c) Suppose that $\widetilde{\mu}$ is σ-finite. Prove (b) without the hypothesis that $\mu^*(E) < +\infty$.

(7) (a) Let m be the Lebesgue measure on $[0, \infty)$, and let μ be a positive measure that is differentiable with respect to m. If μ has a continuous Radon-Nikodym derivative $\dfrac{d\mu}{dm}$, show that the function $x \to \mu([0, x])$ is differentiable, and

$$\frac{d}{dx}\mu([0, x]) = \frac{d\mu}{dm}(x) \text{ for all } x.$$

(b) Let $f : X \to [0, \infty]$ be measurable and let μ be a measure. Show that m_f defined by $m_f(E) := \int_X \chi_E f d\mu$ is a measure and that for any measurable $g : X \to [0, \infty]$ we have $\int_X g dm_f = \int_X g f dm$. (*Hint:* Consider using the Monotone Convergence Theorem.)

(8) Let $\nu : \mathscr{B}_R \to (-\infty, \infty)$ be the finite signed measure given by

$$\nu(E) := \int_{E \cap (-\infty, 0]} \frac{x-2}{x^4+1} dx + \int_{E \cap [0, \infty)} e^{-x} dx + a\delta_{-2}(E),$$

for all $E \in \mathscr{B}_{\mathbb{R}}$. Here $a \geq 0$.

(a) Provide a Hahn Decomposition of $\mathbb{R}$ with respect to ν assuming that $a > 0$.

(b) Provide the Jordan Decomposition of ν assuming that $a > 0$.

(c) Identify $|\nu|$ and provide its Lebesgue Decomposition with respect to the Lebesgue measure m assuming that $a > 0$.

(d) Assume that $a = 0$, compute $\dfrac{d\nu}{dm}$.