**Math 932-933, Comprehensive Exam (first half)** January, 2004

1. State the Picard–Lindelof theorem and then maximize the $\alpha$ in that theorem by choosing the appropriate rectangle concerning the solution of the IVP

$$x' = t + x^2, \quad x(0) = 1.$$

2. State and prove the contraction mapping theorem.

3. Approximate the solution of the IVP

$$x' = \frac{1}{1+x^2}, \quad x(0) = 0$$

by finding the second Picard iterate $x_2(t)$ and find an estimate for how good an approximation it is.

4. Let x(t;a,b) denote the solution of the IVP

$$x' = \arctan x, \quad x(a) = b.$$

Find

$$\frac{\partial x}{\partial b}(t; 0, 0)$$

and

$$\frac{\partial x}{\partial a}(t; 0, 0).$$

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**Instructions:** Do # 4, #5, #6, and 2 more from #1, #2, and #3.

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**1(10pts)** Let $\gamma$ be a periodic orbit of a differentiable dynamical system $\dot{x} = f(x), x \in \mathbb{R}^n$. Let $\Sigma$ be a plane perpendicular to $\gamma$ at a point $p \in \gamma$. If defined, let $P : \Sigma \to \Sigma$ with $x \mapsto P(x) = \phi^{\tau(x)}(x)$, where $\tau$ is the first connecting time from $\Sigma$ to $\Sigma$ following the flow $\phi^t$ of the equation. Prove that $P$ is a local diffeomorphism.

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**2(10pts)** Let $\phi^t$ denote the solution operator for a continuous dynamical system $\dot{x} = f(x), x \in \mathbb{R}^n$. Let $\omega(x) = \{y \in \mathbb{R}^n : \exists t_k \to \infty \text{ s.t. } \phi^{t_k}(x) \to y \text{ as } k \to \infty\}$ be the omega-limit set. Show that $\omega(x)$ is invariant, i.e., $\phi^t(\omega(x)) = \omega(x)$ for any $t$.

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**3(10pts)** Consider the equations with a real parameter $\mu$.

$$\begin{cases} \dot{x} = -x + \mu y^2 \\ \dot{y} = y - x^2 \end{cases}$$

Approximate the parameterized stable manifold at the origin $(0, 0)$ to the 2nd order.

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**4(10pts)** Consider 2 systems of equations

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x \quad \text{and} \quad \dot{y} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} y$$

(a) Construct an explicit conjugacy $h : \mathbb{R}^2 \to \mathbb{R}^2$ between them.

(b) Explicitly verify that the mapping $h$ you have found above is in fact a conjugacy.

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**5(10pts)** STATE and PROVE the Stable Manifold Theorem for the system of equations

$$\begin{cases} \dot{x} = Ax + f(x, y) \\ \dot{y} = By + g(x, y), \end{cases} \quad x \in \mathbb{R}^n, y \in \mathbb{R}^m,$$

if $f, g$ are uniform Lipschitzian.

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**6(10pts)** Consider the system of equations with parameter $0 < \alpha < 1$:

$$\begin{cases} \dot{x} = x(1 - x) - y \\ \dot{y} = (x - \alpha) + (x - \alpha)^2 \end{cases}$$

(a) Find the parameter value $\alpha = \alpha_0$ and the corresponding equilibrium point branch $(x_\alpha, y_\alpha)$ such that the system at $\alpha = \alpha_0$ satisfies the eigenvalue condition of Hopf bifurcation.

(b) Use a sequence of transformations to change the system into the following polar coordinate form

$$\begin{cases} \dot{r} = \mu r + r^2 C_3(\theta, \mu) + r^3 C_4(\theta, \mu) + \dots \\ \dot{\theta} = \beta(\mu) + r D_3(\theta, \mu) + \dots \end{cases}$$

(*Hint:* use $\mu = \alpha - \alpha_0, x := x - x_\alpha, y := y - y_\alpha$ as the first transformation, and find TWO more transformations of your own to complete the process.)

(c) The Hopf Bifurcation Theorem states that if

$$K := \frac{1}{2\pi} \int_0^{2\pi} [C_4(\theta, 0) - \frac{1}{\beta(0)} C_3(\theta, 0) D_3(\theta, 0)] d\theta \neq 0$$

then a periodic orbit must bifurcate from the equilibrium point at $\mu = 0$. Determine the stability of the periodic orbit. (You may need this trigonometric identity: $\int \cos^n \theta d\theta = \frac{1}{n} \cos^{n-1} \theta \sin \theta + \frac{n-1}{n} \int \cos^{n-2} \theta d\theta$.)

(d) Summarize your analysis with a local bifurcation diagram near $\mu = 0$.

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**The End**