Math 932-933 Comprehensive Exam June 2007

1. Show that the integral equation

$$x(t) = e^{2t}\sin t + \frac{1}{2}\int_0^t sx(s)ds$$

has a unique continuous solution on $[0, \sqrt{3}]$.

2. Show that the set of all real sequences $x = \{x_n\}$ with $\lim_{n\to\infty} x_n = 0$ with the sup norm $\|x\| := \sup\{|x_n| : n = 1, 2, 3, \cdots\}$ is a Banach space.

3. State the theorem concerning lower and upper solutions of $x'' = f(t, x)$. Reproduce the part of the proof that if $x(t)$ is a solution of the modified BVP

$$x'' = F(t, x), \quad x(a) = A, \quad x(b) = B,$$

then $x(t) \geq \alpha(t)$ on $[a, b]$.

4. Let $x(t; a, b)$ denote the solution of the IVP

$$x' = \arctan x, \quad x(a) = b.$$

Without solving this IVP, find

$$\frac{\partial x}{\partial b}(t, 0, 0) \quad \text{and} \quad \frac{\partial x}{\partial a}(t, 0, 0).$$

5. State and prove the Ascoli-Arzela Theorem for real-valued functions of a real variable. Give three examples where if you omit one of the three assumptions in the Ascoli-Arzela Theorem, then the conclusion of this theorem is not true.

6. Define the Hilbert projective metric d and prove the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z) \quad \text{for all } \ x, y, z \in P^0.$$

7. Define P is a cone and normal cone. State a theorem concerning equivalent statements to P is a normal cone. Prove any two implications (one way) in your theorem.

8. Prove the following theorem: Theorem. Assume $f : [a, b] \times [0, \infty) \to [0, \infty)$ is continuous and

$$\lim_{x\to 0+} \frac{f(t, x)}{x} = \infty, \qquad \lim_{x\to\infty} \frac{f(t, x)}{x} = 0$$

uniformly for $t \in [a, b]$. Then the BVP

$$-x'' = f(t, x), \quad x(a) = 0, \quad x(b) = 0$$

has a positive solution.