Comprehensive Exam in Ordinary Differential Equations (MATH 932-933)
May 31, 2013

All problems have equal weight.

1. Consider the initial value problem
$$y' = y + \sin(y), \quad y(t_0) = y_0 \in \mathbb{R}$$
Recall that a solution $y(t)$ for t near t_0 can be represented as the fixed-point of some operator T on a complete metric space X. (*Essentially it would be proving a version of the Picard-Lindelöf theorem through a fixed-point method*).

   Specify what T and X are and show that T is a contraction mapping.

2. Let $f(t,y)$ be a continuous function on $E = \{(t,y) : |t-t_0| < a, |y-y_0| < b\}$. Suppose $y' = f(t,y)$, $y(0) = y_0 \in \mathbb{R}^d$ has a solution defined on the interval $[0,\omega)$ where $\omega < \infty$.

   (a) Assume there exists a compact set $K \subset E$ such that $\{(t, y(t))\} \in K$ for all $t \in [0,\omega)$. Prove that the solution can be extended past $t = \omega$. *Hint: first show that* $\lim_{t\to\omega^-} y(t)$ *and* $\lim_{t\to\omega^-} y'(t)$ *exist.*

   (b) Assume $f(t,y)$ is continuous everywhere in $\mathbb{R} \times \mathbb{R}^d$ and that $[0,\omega)$ is a *maximal* right-interval of existence. What can you say about the solution as $t \to \omega^-$?

3. Suppose $y' = f(y)$ and $y(0) = y_0 \in \mathbb{R}^d$. Assume f is Lipschitz continuous. Use Gronwall's inequality to prove directly that a solution to such an ODE must be unique.

4. Let A be a 3×3 real matrix whose Jordan canonical form is $J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and the similarity transformation is S, i.e. $A = S^{-1}JS$. Consider the ODE
$$y' = Ay.$$

   (a) In terms of S provide the exact description of the stable and unstable subspaces of initial data for this problem.

   (b) Prove (briefly and without excessive detail) that there is $\omega > 0$ such that $\lim_{t\to\infty} e^{-\omega t} y(t) = 0$ for every $y_0 \in \mathbb{R}^3$.

   (c) Now consider a perturbed equation $y' = Ay + F(y)$ for a continuous function F. Use the variation of parameter formula to write down the identity that $y(t)$ must satisfy.

5. Consider the equation
$$\ddot{x} + \dot{x}^3 + x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

   (a) Rewrite the equation as a 1st-order system $y' = f(y)$, $y(0) = y_0$ on an appropriate space.

   (b) Show that this system is stable in the sense of Lyapunov by constructing (with proof) a Lyapunov function V. *Hint: to determine a Lyapunov function consider the scalar product of position and velocity vectors in the first-order formulation.*

   (c) What are the invariant subsets of the set where $\dot{V}(y) := \nabla V \cdot f(y) = 0$?

   (d) What can you say about the *asymptotic* Lyapunov stability of this system?

6. Suppose y takes values in $\mathbb{R}_+ = [0,\infty)$ and satisfies the equation $y' = -y^2$.

(a) Find the dynamical system (X, S_t) generated by this problem. Show that the family of evolution operators $\{S_t\}$ has the requisite properties *(don't forget to briefly mention the continuity requirements)*.

(b) Prove that this system is dissipative.

7. Suppose a dynamical system $(\mathbb{R}^d, S_t)$ has a bounded absorbing set B. Prove *directly* that $\omega(B)$ is negatively invariant.

8. Assume a dynamical system (X, S_t) is dissipative and asymptotically compact. *Hint: only short concise answers are expected for each of the questions below.*

   (a) Must there be a global attractor for this system? If so, how can it be obtained?

   (b) Let a be a point in the global attractor $\mathcal{A}$. Prove that there exists a (full) trajectory that passes through a.

   (c) Suppose in addition there exists a Lyapunov function (in the sense of the textbook by I. Chueshov) on X. Give the most precise description of the attractor that you can in this setting and provide a schematic picture of it.