Ph.D. Qualifying Exam and Masters Comprehensive Exam
Algebra: 817–818. January 24, 1998.

Do exactly **two** problems from each section (for a total of **six** problems). If you work on more than six problems, or on more than two from any one section, clearly indicate which you want graded. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section I: Groups

1. Let G be a non-abelian group of order 21. Show that G is generated by elements x and y satisfying the relations $y^{-1}xy = x^2$. (Hint: First show that $y^{-1}xy$ must be either x^2 or x^4. In the latter case replace y by its inverse.)
2. Let G be a group of order 30. Prove that G has a normal subgroup of order 15.
3. Let M and N be normal subgroups of G such that $G = MN$. Prove that
$$G/(M \cap N) \cong (G/M) \times (G/N).$$
Note: Formal manipulation of isomorphism theorems will not work here; you'll actually need to explicitly define a map.
4. Let $\sigma = (12345) \in S_5$, and let $H = \langle \sigma \rangle$ (the cyclic subgroup of S_5 generated by σ). Find the order of the normalizer $N_{S_5}(H)$ of H in S_5.

Section II: Rings and Fields

5. Prove that $5x^4 + 7x^3 + 11x^2 + 6x + 1$ is irreducible in $\mathbb{Q}[x]$. Hint: Think about how the polynomial factors modulo 2.
6. Let R be an integral domain (commutative with 1).
   (a) Define "irreducible element" of R.
   (b) Define "prime element" of R.
   (c) Prove that every prime element of R is irreducible.
   (d) If every non-zero non-unit of R can be expressed as a product of prime elements, prove that R is a unique factorization domain.
7. (a) Let R be a ring with 1 having elements x and y such that $xy = 1$ but $yx \neq 1$.
      i. Prove that $Rx \neq R$.
      ii. Prove that $R = Rx \oplus R(1 - yx)$.
      iii. Show that the ring of linear transformations on a countably infinite dimensional vector space has elements x and y such that $xy = 1$ but $yx \neq 1$.
   (b) Prove that the polynomial $x^7 + 7x + 7$ is irreducible over the field $\mathbb{Q}(\sqrt[5]{5})$. (Hint: Think about degrees of field extensions.)

Section III: Linear Algebra

(c) Let W be a vector space of finite dimension d, let U and V be subspaces of dimensions r and s, respectively. Prove that $\dim(U \cap V) \geq r + s - d$. (You should do this pretty much from scratch, by choosing bases for various relevant vector spaces.)

(d) Let A and B be matrices with real entries, and assume that AB is defined. Prove that $\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

(e) Let A be an 11×11 matrix with list of elementary divisors
$$x^2, x^3, x-2, x+3, x+3, (x+3)^3.$$
   i. Write down the list of invariant factors for A.
   ii. Find the minimal and characteristic polynomials of A over $\mathbb{Q}$.
   iii. Find the Jordan canonical form of A.
   iv. Find the rational canonical form of A.

(f) Let V be a vector space over a field F, let S be a set that spans V, and let T be a linearly independent subset of V. Prove that there is a subset S_0 of S such that $S_0 \cap T = \emptyset$ and $S_0 \cup T$ is a basis for V. (Do not assume that V is finite dimensional. You will need to use Zorn's lemma or the equivalent.)