Master's Comprehensive and Ph.D. Qualifying Exam
Algebra: Math 817-818, January 30, 1999

Do 6 problems, 2 from each of the three sections. If you work on more than six problems, or on more than 2 from any section, clearly indicate which you want graded. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section I: Groups

[I.1] Let S be the set of all positive integers n such that every group of order 75 contains an element of order n. Determine S and justify your answer.

[I.2] Let G be a group of order $(p \cdot q)^2$, where p and q are primes with $p > 3$ and $q = p + 2$. Prove that G is either cyclic or isomorphic to the product of two cyclic groups.

[I.3] Let G be a finite group with $x \in G$ such that $|G| = nm$.
(a) Suppose $x \in G$ has order n and let $\sigma_x \in S_G$ be the permutation such that $\sigma_x(g) = xg$ for every $g \in G$. Show that σ_x is a product of m disjoint n-cycles.
(b) If $n = 2$ and m is odd, show that there is a homomorphism $f : G \to S_G$ such that $f(G)$ contains an odd permutation.
(c) If $n = 2$ and m is odd, conclude that G has a subgroup of index 2.

[I.4] Let G be a group. Recall that the *commutator subgroup* of G is defined to be the subgroup G' of G generated by the set of all elements of the form $x^{-1}y^{-1}xy$, where x and y are elements of G.
(a) Show that G' is a normal subgroup of G.
(b) Show that G/G' is abelian.
(c) For any normal subgroup N of G, show that G/N is abelian if and only if $G' \leq N$.

Section II: Rings and Fields

[II.5] Prove that $3x^5 + 5x^4 + 9x^3 + 6x^2 + x + 7$ is irreducible in $\mathbf{Q}[x]$, where $\mathbf{Q}$ is the field of rational numbers.

[II.6] Let R be the ring of all continuous real-valued functions on the unit interval. Let $c \in [0, 1]$ and denote $\{f \in R : f(c) = 0\}$ by M_c.
(a) Show that M_c is a maximal ideal of R.
(b) Show that every maximal ideal of R is of the form M_c for some $c \in [0, 1]$. Hint: Use compactness.

[II.7] Let A be a finite commutative ring with $1 \neq 0$. Show that every prime ideal of A is maximal.

[II.8] Prove that the group of invertible elements of a finite field is cyclic.

Section III: Modules and Vector Spaces

[III.9] Let $\mathbf{C}$ be the field of complex numbers and let x be an indeterminate. Let $L : \mathbf{C}[x]/(x^2(x-1)^3) \to \mathbf{C}[x]/(x^2(x-1)^3)$ be the linear transformation given by multiplying by x.
(a) Find the Jordan Canonical Form for L.
(b) Find a basis of $\mathbf{C}[x]/(x^2(x-1)^3)$ with respect to which the matrix for L is the Jordan Canonical Form.

[III.10] Let V be a vector space. Do not assume that V is finite dimensional. Let $W \subset V$ be a spanning set for V. Show that W contains a basis for V.

[III.11] Let V be a finite dimensional vector space. Let $L : V \to V$ be a linear transformation. Show that there is a positive integer n such that $(\ker L^n) \cap (\operatorname{Im} L^n) = \{0\}$.

[III.12] Let $G = GL(2, \mathbf{Q})$ be the group of invertible 2×2 matrices over the rational numbers. Determine up to similarity all elements of G which have order exactly 4.