**Master's Comprehensive and Ph.D. Qualifying Exam**
**Algebra: Math 817-818, January 27, 2000**

Do 6 problems, 2 from each of the three sections. If you work on more than six problems, or on more than 2 from any section, clearly indicate which you want graded. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Note: $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real and complex numbers respectively, and $\mathbb{Z}$ denotes the ring of integers. Problems marked with an asterisk deal with material not generally covered in the old Math 817–818 syllabus.

## Section I: Groups

(1) Let $\mathrm{A}_5$ denote the alternating group of even permutations of $\{1, 2, 3, 4, 5\}$.

(a) Prove that every element of $\mathrm{A}_5$ has prime-power order, and deduce that $\mathrm{A}_5$ is the union of its Sylow subgroups.

(b) For each prime $p \leq 5$, determine the number of Sylow $p$-subgroups of $\mathrm{A}_5$.

(2) Let $G$ be a finite group in which the order of every element is a prime power. Show that either the order of $G$ is a prime power or the center of $G$ is trivial.

(3) Show that there is no simple group of order $10,000 = 10^4$.

(*4) Let $n$ be an arbitrary positive integer.

(a) Show that for every real $n \times n$ matrix $A$ there is a positive real number $\epsilon$ such that $tI_n + A$ is invertible for all $t$ with $0 < t < \epsilon$.

(b) Show that an element $M$ of the center of $\mathrm{GL}_n(\mathbb{R})$ commutes with every real $n \times n$ matrix.

(c) Prove that the center of $\mathrm{GL}_n(\mathbb{R})$ consists precisely of the scalar matrices. (You might consider the consequences of a matrix's commuting with the matrix $E_{ij}$, which has a 1 in the $ij$ position and 0s elsewhere.)

(*5) Let $G$ be a finite subgroup of the group of isometries (rigid motions) of the plane. Prove that $G$ has a fixed point (that is, there is a point $p$ in the plane such that $g(p) = p$ for every $g \in G$).

## Section II: Rings and Fields

(6) Let $F$ be a field and let $p_1(x), \ldots, p_r(x)$ be distinct monic irreducible polynomials in $F[x]$. Let $f(x) = p_1(x)^{n_1} \cdots p_r(x)^{n_r}$, where the $n_i$ are positive integers.

(a) Determine the number of ideals in $F[x]/(f(x))$. Justify your answer.

(b) Determine the number of prime ideals in $F[x]/(f(x))$. Justify your answer.

(7) Let $f(x) = x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 7$. Prove that $f(x)$ is irreducible over the rational numbers. Hint: You might think about how $f(x)$ factors over $\mathbb{Z}/(2)$.

(8) Let $R = \mathbb{Z}[i]$, the ring of Gaussian integers. Given $\alpha = a + bi \in R$ (with $a, b \in \mathbb{Z}$), define the "norm" of $\alpha$ by $\mathcal{N}(\alpha) = |\alpha|^2 = a^2 + b^2$.

(a) Prove the "division algorithm": Given $\alpha, \beta \in R$ with $\alpha \neq 0$, there exist $\gamma, \rho \in R$ such that $\beta = \gamma\alpha + \rho$ and $\mathcal{N}(\rho) < \mathcal{N}(\alpha)$. (Hint: First do the division in $\mathbb{C}$.)

(b) Use (a) to prove that $R$ is a principal ideal domain.

(9) Let $p$ and $q$ be distinct prime numbers. Prove that $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$ and that $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ has degree 4 over $\mathbb{Q}$.

## Section III: Linear Algebra

(10) Find the Jordan canonical form and the rational canonical form of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ over the field $\mathbb{F}_3$ with 3 elements.

(11) Let $V$ be the real vector space $\mathbb{R}[x]/(x^3(x-1))$. Let $L : V \to V$ be the linear operator given by multiplication by $x$.

(a) Find the Jordan canonical form for $L$.

(b) Find the rational canonical form for $L$.

(c) Find the minimal polynomial for $L$. Explain and justify everything.

(*12) Let $A$ and $P$ be $n \times n$ matrices over $\mathbb{C}$.

(a) Define what it means to say that $A$ is a *Hermitian* matrix.

(b) Define what it means to say that $P$ is a *unitary* matrix.

(c) Prove that if $A$ is Hermitian there is a unitary matrix $P$ such that $PAP^{-1}$ is a real diagonal matrix. (Your proof should proceed pretty much from the definition and not use any fancy results on diagonalization. You should not, for example, use the fact that a normal matrix is unitarily diagonalizable!)

(13) Let $V$ be a vector space (not necessarily finite-dimensional) over a field $F$. Let $S$ be a linearly independent subset of $V$. Let $T$ be a subset of $V$ such that the span of $T$ is $V$. Prove that $V$ has a basis $B$ such that $S \subseteq B \subseteq S \cup T$.