**Master's Exam and Ph.D. Qualifying Exam**
**Algebra: Math 817-818, January 18, 2001**

Do 6 problems, 3 from each of the two sections. If you work on more than six problems, or on more than 3 from any section, clearly indicate which you want graded. For problems with more than one part, do not assume that all parts count equally. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Note: $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real and complex numbers respectively, and $\mathbb{Z}$ denotes the ring of integers.

## Section I: Groups and Linear Algebra

(1) Let $H$ be a proper subgroup of a finite group $G$. Prove that the union of the conjugates of $H$ is not all of $G$.

(2) Recall that $\mathrm{O}_n$ denotes the orthogonal group of $n \times n$ real matrices $\alpha$ satisfying $\alpha\alpha^{\mathrm{t}} = \mathrm{I}_n$ and that $\mathrm{SO}_n$ is the *special orthogonal group*, which consists of matrices of determinant 1 in $O_n$. (Note: $\mathrm{I}_n$ denotes the $n \times n$ identity matrix, and $\alpha^{\mathrm{t}}$ is the transpose of $\alpha$.)
  (a) Prove that $\mathrm{SO}_n$ is a subgroup of index 2 in $\mathrm{O}_n$.
  (b) Prove that $\mathrm{O}_3$ is the internal direct product of $\mathrm{SO}_3$ and $\{\pm\mathrm{I}_3\}$.
  (c) Prove that there is no normal subgroup $N$ of $\mathrm{O}_2$ such that $\mathrm{O}_2$ is the internal direct product of $\mathrm{SO}_2$ and $N$.

(3) Let $\mathbb{F} = \mathbb{F}_3$ denote the field with 3 elements, and let $V = \mathbb{F}^2$. Let $\alpha, \beta, \gamma$ and $\delta$ be the four one-dimensional subspaces of $V$ (each having three elements), spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. Let $G := \mathrm{GL}_2(\mathbb{F})$ be the group of $2 \times 2$ invertible matrices over $\mathbb{F}$, and let $G$ act on $\{\alpha, \beta, \gamma, \delta\}$ by matrix multiplication.
  (a) Prove that the kernel of the homomorphism $\Phi : G \to \mathrm{S}_4$ corresponding to this action is $\{\pm\mathrm{I}_2\}$. (Note: $\mathrm{I}_2$ denotes the $2 \times 2$ identity matrix.)
  (b) Prove that $\Phi$ is surjective, and deduce that $G/\{\pm\mathrm{I}\} \cong \mathrm{S}_4$.

(4) List all possible rational canonical forms (over $\mathbb{Q}$) and Jordan canonical forms (over $\mathbb{C}$) for $8 \times 8$ matrices having determinant 81 and minimal polynomial $(x-3)^2(x^2+1)$. Justify your reasoning.

(5) Recall that the cokernel of an $m \times n$ integer matrix $A$ is $\mathbb{Z}^m/C$, where $C$ is the "column space" of $A$, that is, the subgroup of $\mathbb{Z}^m$ generated by the columns of $A$. For each of the matrices

$$A := \begin{bmatrix} 3 & 8 & 7 & 9 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 2 & 1 \end{bmatrix} \qquad B := \begin{bmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{bmatrix} \qquad C := \begin{bmatrix} 4 & 2 \\ 7 & 4 \\ 2 & 2 \end{bmatrix},$$

  (a) reduce the matrix to diagonal form by integer row and column operations, and
  (b) Express the cokernel of the matrix as a direct sum of cyclic groups.

## Section II: Rings, Fields, Modules

(6) Let $\alpha$ and $\beta$ be complex numbers of degree 3 over $\mathbb{Q}$.

(a) Show that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$ is $3, 6$ or $9$.

(b) Give examples showing that all three possibilities can occur, and prove your assertion in the case of degree 6.

(7) Let $f$ and $g$ be non-zero polynomials with integer coefficients. Prove that $f$ and $g$ are relatively prime in $\mathbb{Q}[x]$ if and only if the ideal of $\mathbb{Z}[x]$ generated by $f$ and $g$ contains a non-zero constant.

(8) Let $I$ be a non-zero ideal of a commutative ring $R$ with identity. Prove that $I$ is a free $R$-module if and only if $I = Ra$ for some non-zerodivisor $a \in R$.

(9) Let $R$ be an integral domain (commutative with identity).

(a) Let $a, b, c \in R$. Define *precisely* what it means to say that $c$ is a *least common multiple* of $a$ and $b$.

(b) Find two elements of $R := \mathbb{Z}[\sqrt{-10}]$ that do not have a least common multiple. (You may assume standard properties of the norm N (the square of the usual complex modulus): $\mathrm{N}(r + s\sqrt{-10}) = r^2 + 10s^2$, for $r, s \in \mathbb{Z}$.)

(10) Let $\mathbb{F}$ be a field with 81 elements. Prove that every element of $\mathbb{F}$ has a unique $9^{\text{th}}$ root in $\mathbb{F}$. (That is, $\forall \alpha \in \mathbb{F}$, $\exists!\ \beta \in \mathbb{F}$ such that $\beta^9 = \alpha$.)