Master's Comprehensive and Ph.D. Qualifying Exam
Algebra: Math 817-818, January 19, 2002

Do 6 problems, 2 from each of the three sections. If you work on more than six problems, or on more than 2 from any section, clearly indicate which you want graded. Different parts of a problem do not necessarily count the same.

Justify everything carefully. You may quote and use well-known theorems, provided they do not make the problem trivial. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem or appeal to known results in such a way that the problem becomes trivial.

You should have no need for a calculator on this exam, but you may, if you wish, use your calculator for routine computations with numbers. You may not use any linear algebra software that might be installed on your calculator.

Note: $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real and complex numbers respectively. The ring of integers is denoted by $\mathbb{Z}$, $\mathbb{N}$ is the set of positive integers, and C_n is the cyclic group of order n.

Section I: Groups and Geometry

1. Let G be a group (not necessarily finite). A subgroup S of G is said to be *characteristic* provided $\sigma(S) \subseteq S$ for every $\sigma \in \mathrm{Aut}(G)$.

(a) Prove that every characteristic subgroup of G is a normal subgroup of G.

(b) Prove that the center of a group G is a characteristic subgroup of G.

(c) Prove that if S is a characteristic subgroup of G then $\sigma(S) = S$ for every $\sigma \in \mathrm{Aut}(G)$.

(d) Let p be a prime and let P be the subgroup of G generated by all elements of G whose order is a power of p. Prove that P is a characteristic subgroup of G.

2. Group actions

(a) Let G be a group of order 15 acting on a set with 7 elements. Prove that there exists at least one fixed point.

(b) Give an example of an action of C_{15} on a set with 8 elements with no fixed points. Justify.

3. Rigid Motions of the Plane: Let τ be a translation of the plane, and let ρ be a non-trivial rotation about some point in the plane.

(a) Prove that $\rho\tau$ has a fixed point. (τ acts first.)

(b) Prove that $\tau\rho$ has a fixed point.

Give direct and rigorous proofs, without assuming the classification of symmetries of the plane.

4. Dense subgroups of $\mathbb{R}^+$.

(a) Let G be a subgroup of the additive group $\mathbb{R}$ containing arbitrarily small positive real numbers. Prove that G is dense in $\mathbb{R}$. (That is, given real numbers a and b with $a < b$, prove that there is an element $g \in G$ with $a < g < b$.)

(b) Prove that $G := \mathbb{Z} + \mathbb{Z}\sqrt{2}$, the additive subgroup of $\mathbb{R}$ generated by 1 and $\sqrt{2}$, is dense in $\mathbb{R}$. (One approach is to show that G does not have a smallest positive element and to deduce (a) from this fact. Other methods are possible.)

Section II: Rings and Fields

5. Field Extensions

(a) Give an example of a field extension L/F of finite degree having two distinct intermediate fields K_1 and K_2 such that K_1 and K_2 are isomorphic as fields. Justify.

(b) Suppose K_1 and K_2 are two distinct subfields of $\mathbb{C}$ such that $[K_1 : \mathbb{Q}] = [K_2 : \mathbb{Q}] = 2$. Prove that K_1 and K_2 are not isomorphic fields. (You may assume that every quadratic extension of $\mathbb{Q}$ is of the form $\mathbb{Q}(\sqrt{d})$ for some square-free integer $d \neq 1$.)

6. Prove that the polynomial $f(x) := x^5 + 6x^4 + 3x^2 + x + 2$ is irreducible over $\mathbb{Q}$. (You might find it helpful to reduce modulo 2.)

7. Let K/k be a field extension (not necessarily algebraic).

(a) Let R be a ring with $k \subseteq R \subseteq K$, and assume that every element of R is algebraic over k. Prove that R is a field.

(b) Let E and F be intermediate fields, both of them algebraic over k. Prove that

$$\{\sum_{i=1}^{n} e_i f_i \mid e_i \in E, f_i \in F, n \in \mathbb{N}\}$$

is the smallest subfield of K containing both E and F.

8. Let R be an integral domain (commutative with identity). Suppose that the polynomial ring $R[x]$ is a principal ideal domain. Prove that R is a field.

Section III: Linear Algebra

9. Similarity

(a) How many similarity classes of 6×6 matrices with characteristic polynomial $(x^2+1)^3$ are there over $\mathbb{C}$? Explain.

(b) How many similarity classes of 6×6 matrices with characteristic polynomial $(x^2+1)^3$ are there over $\mathbb{Q}$? Justify your answer, and give one representative from each class.

10. Recall that the *cokernel* of an $m \times n$ matrix α over $\mathbb{Z}$ is the abelian group $\mathbb{Z}^m/C$, where C is the subgroup of $\mathbb{Z}^m$ generated by the columns of α. For each of the following, reduce the matrix to diagonal form by doing integer row and column operations, and express the cokernel as a direct sum of cyclic groups.

(a) $\alpha = \begin{bmatrix} 6 & 4 & 2 \\ 6 & 2 & 4 \end{bmatrix}$.

(b) $\beta = \begin{bmatrix} 6 & 6 \\ 4 & 2 \\ 2 & 4 \end{bmatrix}$.

11. Let A be an $n \times n$ real matrix. Let $|| \quad ||$ and $\langle \ , \ \rangle$ denote the usual norm and inner product on $\mathbb{R}^n$. (Thus, viewing elements of $\mathbb{R}^n$ as column vectors, we have $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^t \mathbf{w}$ and $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.) Prove that the following conditions (various criteria for A to be orthogonal) are equivalent:

(a) $A^t A = \mathrm{I}_n$.

(b) $||A\mathbf{v}|| = ||\mathbf{v}||$ for each $\mathbf{v} \in \mathbb{R}^n$.

(c) $\langle A\mathbf{v}, A\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

(d) The columns of A form an orthonormal basis for $\mathbb{R}^n$.

12. Let $A = \begin{bmatrix} -1 & -9 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 2 & 7 & 2 & 0 \\ 4 & 13 & 0 & 2 \end{bmatrix}$.

(a) Show that the characteristic polynomial of A is $(\lambda - 2)^4$.

(b) Find the Jordan canonical form J of A. Justify.

(c) We know there is an invertible matrix P such that $PAP^{-1} = J$. Find either P or P^{-1}. (It's your choice, but be sure to clarify which one you are finding!)