**Master's Comprehensive and Ph.D. Qualifying Exam**
**Algebra: Math 817-818, January 21, 2003**

Do 6 problems, 3 from each of the two sections. If you work on more than six problems, or on more than 3 from any section, clearly indicate which you want graded. Different parts of a problem do not necessarily count the same.

Justify everything carefully. You may quote and use well-known theorems, provided they do not make the problem trivial. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem or appeal to known results in such a way that the problem becomes trivial.

You should have no need for a calculator on this exam, but you may, if you wish, use your calculator for routine computations with numbers. You may not use any linear algebra software that might be installed on your calculator.

Note: $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real and complex numbers respectively. The ring of integers is denoted by $\mathbb{Z}$.

## Section I: Groups, Geometry and Linear Algebra

**1.** Let $T$ be a linear operator on a finite-dimensional vector space $V$. Prove that there is a positive integer $n$ such that $V = \operatorname{Im}(T^n) \oplus \operatorname{Ker}(T^n)$. (Hint: Consider the subspaces $\operatorname{Ker}(T) \subseteq \operatorname{Ker}(T^2) \subseteq \operatorname{Ker}(T^3) \subseteq \dots$ and $\operatorname{Im}(T) \supseteq \operatorname{Im}(T^2) \supseteq \operatorname{Im}(T^3) \supseteq \dots$.)

**2.** Let $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (with entries in $\mathbb{C}$). Find the Jordan canonical form $J$ of $A$, and find a matrix $P$ such that $P^{-1}AP = J$.

**3.** Group Actions.

(a) Prove that the alternating group $A_4$ (of order 12) has no subgroup of order 6.

(b) Prove that for every action of $A_4$ on a set $X$ with 5 elements there is an element $x \in X$ that is fixed by every element of $A_4$.

**4.** Assuming the fact that every $\mathbb{C}[x]$-module is a direct sum of cyclic modules, deduce the existence of the Jordan canonical form for a linear operator $T$ on a finite-dimensional complex vector space. (You may also use standard things like the Chinese Remainder Theorem and the fact that $\mathbb{C}$ is algebraically closed.)

**5.** Recall that the *cokernel* $\operatorname{Coker}(\alpha)$ of an $m \times n$ matrix $\alpha$ over $\mathbb{Z}$ is the abelian group $\mathbb{Z}^m/C$, where $C$ is the subgroup of $\mathbb{Z}^m$ generated by the columns of $\alpha$. Prove that if $\sigma$ and $\tau$ are invertible matrices over $\mathbb{Z}$ ($m \times m$ and $n \times n$ respectively), then $\operatorname{Coker}(\sigma\alpha\tau)$ is isomorphic to $\operatorname{Coker}(\alpha)$.

**6.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function. Prove that the following conditions are equivalent:

(a) $f$ is a rigid motion (i.e., an isometry) fixing 0.

(b) $f$ is left multiplication by an orthogonal matrix.

## Section II: Rings, Fields and Modules

**7.** Find a greatest common divisor of $18+\mathrm{i}$ and $11+7\mathrm{i}$ in the ring $\mathbb{Z}[\mathrm{i}]$ of Gaussian integers, and verify that it is indeed a greatest common divisor.

**8.** Prove that the polynomial $f(x) := \frac{1}{5}x^5 + \frac{1}{3}x^4 + x^3 + \frac{2}{3}x^2 + \frac{1}{5}x + \frac{1}{3}$ is irreducible over $\mathbb{Q}$.

**9.** For a module $M$ over a commutative integral domain $R$, the *torsion* submodule of $M$ is the set $M_{\text{tors}} := \{x \in M \mid cx = 0 \text{ for some non-zero } c \in R\}$. The module $M$ is said to be *torsion-free* provided $M_{\text{tors}} = \{0\}$.

(a) Prove that $M_{\text{tors}}$ is a submodule of $M$.

(b) Prove that $M/M_{\text{tors}}$ is torsion-free.

(c) Prove that if $N$ is a submodule of $M$ such that both $N$ and $M/N$ are torsion-free, then $M$ is torsion-free.

**10.** Wilson's Theorem

(a) Let $G$ be a finite subgroup of the multiplicative group $\mathbb{F}^\times$ of a field $\mathbb{F}$ of characteristic different from 2, and let $\alpha$ be the product of the elements of $G$. Prove that $\alpha = 1$ if $G$ has odd order, and that $\alpha = -1$ if $G$ has even order.

(b) Prove that $(p-1)! \equiv -1 \pmod{p}$ for each prime number $p$.

**11.** Let $p$ and $q$ be distinct (positive) prime integers, and put $\alpha := \sqrt{p} + \sqrt{q}$. Let $K = \mathbb{Q}(\alpha)$. Prove in detail that $[K : \mathbb{Q}] = 4$.

**12.** Let $R$ be a ring with 1, let $L$ be a minimal non-zero left ideal of $R$, and let $r \in R$. Prove that if $Lr \neq 0$ then $Lr$ is a minimal left ideal of $R$. (Hint: If $Lr \neq 0$, show $Lr$ is isomorphic to $L$ as a left $R$-module.)