Math 817–818 Qualifying Exams and Masters Comprehensive Exam

January 20, 2004 2:00–5:00PM

- Do *two* of the four given problems from each of the three sections, for a total of *six* problems. Be sure to make it clear which six problems you want graded.
- If you have doubts about the wording of a problem or which results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.
- Be sure to show your reasoning clearly and explain everything carefully.

I. Groups and Geometry

1. Suppose H is a normal subgroup of a group G and $\phi : G \to H$ is a group homomorphism so that $\phi(h) = h$ for all $h \in H$. Prove $G \cong H \times \ker(\phi)$. Your proof should include an explicit definition of the isomorphism.
2. Suppose that N is a normal subgroup of a finite group G, that $|N| = 7$, that $2 \nmid |G|$, and that $3 \nmid |G|$. Prove N is contained in the center of G.
3. Prove $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$ (where $\mathbb{F}_2$ is the field with two elements and S_3 is the symmetric group on three symbols). You may *not* use, without proof, the fact that there is only one non-abelian group of order 6 up to isomorphism. *Hint*: Begin by defining an action of $\mathrm{GL}_2(\mathbb{F}_2)$ and the set of non-zero vectors in the vector space $\mathbb{F}_2^2$.
4. Let f and g be orientation preserving rigid motions of the plane.
   (a) Prove $fgf^{-1}g^{-1}$ is a translation (possibly the trivial one).
   (b) Let G be a subgroup of the group M of all rigid motions of the plane. Prove that if G contains non-trivial rotations about different points, then G contains a non-trivial translation.

II. Linear Algebra

5. Let M be the $\mathbb{C}[t]$-module $\dfrac{\mathbb{C}[t]}{t(t^2-1)} \oplus \dfrac{\mathbb{C}[t]}{(t+1)^3}$. Define V to be the complex vector space underlying M (that is, V is M with the action of t forgotten) and let θ be the linear endomorphism on V defined by multiplication by t on M.
   (a) Find, with proof, the Jordon Canonical Form of θ.
   (b) Find, with proof, the Rational Canonical Form of θ.
6. Let A be an $n \times n$ Hermitian complex matrix. Prove $A = 0$ if and only if $\mathrm{trace}(A^2) = 0$.
7. Prove the only $n \times n$ real matrix that is symmetric, orthogonal, and positive definite is I_n.
8. Let A be an $n \times n$ complex matrix. Prove that for every $\epsilon > 0$, there is a matrix that is diagonalizable and "within ϵ" of A – that is, prove there is an $n \times n$ complex matrix B such that B is diagonalizable and $\|b_{i,j} - a_{i,j}\| < \epsilon$, for all $1 \leq i, j \leq n$, where $a_{i,j}$ and $b_{i,j}$ denote the (i, j)-entries of A and B. *Hint*: One proof uses that A is similar to an upper-triangular matrix and that a matrix is diagonalizable if its eigenvalues are distinct.

III. Rings, Fields, and Modules

9. Let R be a commutative ring and let I and J be ideals of R. Prove that if $I + J = R$ and $I \cdot J = \{0\}$, then $R \cong R/I \times R/J$ as rings.

10. Let $R = \mathbb{Z}[\sqrt{-5}]$.

    (a) Find, with proof, elements $x, y \in R$ that do *not* have a GCD. *Hint*: Note that $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$

    (b) Find, with proof, an ideal of R that is *not* principal.

11. Prove that $f(x) = \frac{1}{2}x^3 + 3x^2 - \frac{5}{3}x - 5$ is irreducible over the field $\mathbb{Q}(\sqrt[4]{5})$.

12. The ring $\mathbb{Z}[\sqrt{-2}]$ is a UFD (a fact you need not prove). Let p be a positive prime integer. Prove that the following conditions are equivalent.

    (a) p is a prime element of $\mathbb{Z}[\sqrt{-2}]$.

    (b) $x^2 + 2$ is irreducible in $\mathbb{F}_p[x]$ (where $\mathbb{F}_p = \mathbb{Z}/p$).

    (c) p cannot be written as $a^2 + 2b^2$ for any integers a and b.