Master's Comprehensive and Ph.D. Qualifying Exam
Algebra: Math 817-818, January 18, 2005

Do 6 problems, 2 from each of the three sections. If you work on more than six problems, or on more than 2 from any section, clearly indicate which you want graded. Different parts of a problem do not necessarily count the same.

Justify everything carefully. You may quote and use well-known theorems, provided they do not make the problem trivial. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem or appeal to known results in such a way that the problem becomes trivial.

Do not use calculators or computers on this exam.

Note: $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real and complex numbers respectively. The ring of integers is denoted by $\mathbb{Z}$, $\mathbb{N}$ is the set of positive integers, C_n is the cyclic group of order n and I_n is the $n \times n$ identity matrix.

Section I: Groups and Geometry

1. Let G be a finite group and m a positive integer which is relatively prime to $|G|$. If $b \in G$ and $a^m b = ba^m$ for all $a \in G$, show that b is in the center of G.

2. Let n be a positive integer, and consider the following two subgroups of $G := \mathrm{GL}_n(\mathbb{R})$:

$$H = \mathrm{SL}_n(\mathbb{R}) \quad (= \{g \in G \mid \det(g) = 1\})$$
$$N = \{\alpha \mathrm{I}_n \mid \alpha \in \mathbb{R}^\times\} \quad (= \text{ the group of non-zero scalar matrices})$$

Prove that the following conditions are equivalent:

(a) $H \cap N = \{\mathrm{I}_n\}$.
(b) $HN = G$.
(c) n is odd.

3. Let G be a nonabelian group of order $2p$, where p is an odd prime. For each positive integer n, determine the number of conjugacy classes in G of size n.

4. Let $\mathbb{R}^+$ denote the group of real numbers under addition.

(a) Let G be a subgroup of $\mathbb{R}^+$ containing arbitrarily small positive real numbers. Prove that G is dense in $\mathbb{R}^+$. (That is, given real numbers a and b with $a < b$, prove that there is an element $g \in G$ with $a < g < b$.)

(b) Prove that $\mathbb{Z} + \mathbb{Z}\sqrt{2}$, the subgroup of $\mathbb{R}^+$ generated by 1 and $\sqrt{2}$, is dense in $\mathbb{R}^+$. (One approach is to use (a). Other methods are possible.)

Section II: Linear Algebra

5. Let A be an $n \times n$ matrix all of whose entries are zeros and ones, in a checkerboard pattern. Assume A has 1's on the diagonal. (Note: Part (a) of this problem is probably *not* relevant for Part (b).)

(a) If $n = 4$, prove that A is similar over $\mathbb{Q}$ to $\begin{bmatrix} 2&0&0&0 \\ 0&2&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$.

(b) For any $n \geq 2$, show that there is an invertible real $n \times n$ matrix P such that $P^t AP$ is a diagonal matrix all of whose entries are 0 except for two 1's on the diagonal.

6. Recall that the *cokernel* of an $m \times n$ matrix α over $\mathbb{Z}$ is the abelian group $\mathbb{Z}^m/C$, where C is the subgroup of $\mathbb{Z}^m$ generated by the columns of α. For each of the following, reduce the matrix to diagonal form by doing integer row and column operations, and express the cokernel (up to isomorphism) as a direct sum of cyclic groups:

$$\alpha = \begin{bmatrix} 6 & 10 & 15 \\ 2 & 4 & 6 \\ -2 & 2 & 0 \\ 4 & 6 & 9 \end{bmatrix} \qquad \beta = \alpha^{\mathrm{t}} = \begin{bmatrix} 6 & 2 & -2 & 4 \\ 10 & 4 & 2 & 6 \\ 15 & 6 & 0 & 9 \end{bmatrix}$$

7. Let A be an 8×8 nilpotent matrix over $\mathbb{C}$. Assume $\mathrm{rank}(A) = 5$ and $\mathrm{rank}(A^2) = 2$. List all possible Jordan canonical forms for A, and show that knowledge of $\mathrm{rank}(A^3)$ would allow one to determine the Jordan canonical form of A.

8. Let A be an $n \times n$ matrix over $\mathbb{Q}$. Prove that A is similar to A^{t}.

Section III: Rings and Fields

9. Find, with proof, a polynomial $f(x)$ of degree 3 that is irreducible over the field $\mathbb{F}_{81}$ with 81 elements. (Give an explicit polynomial.)

10. Consider the polynomial rings over the integers and the rationals, $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$.

(a) Let $f(x)$ and $g(x)$ be non-zero polynomials in $\mathbb{Z}[x]$, and let I be the ideal of $\mathbb{Z}[x]$ generated by $f(x)$ and $g(x)$. Prove that $f(x)$ and $g(x)$ are relatively prime in $\mathbb{Q}[x]$ if and only if $I \cap \mathbb{Z} \neq (0)$.

(b) Find with justification an ideal of $\mathbb{Z}[x]$ that is not principal.

11. Let $f \in \mathbb{Z}[x]$, where x is an indeterminate. Prove that $\mathbb{Z}[f] = \mathbb{Z}[x]$ if and only if f is a linear polynomial with leading coefficient ± 1.

12. Let R be a non-zero ring with 1, and assume that the set N of non-invertible elements of R is closed under addition. Prove that N is the unique maximal left ideal of R. (The trickiest part is probably showing that N is a left ideal. The equation $xy = (1 + x)y - y$ might be useful. You must allow for the fact that having a left inverse does not necessarily make an element invertible; that is, in a general ring with 1, $xy = 1$ does not necessarily force x and y to be invertible.)