Master's Comprehensive and Ph.D. Qualifying Exam
Algebra: Math 817-818, January 17, 2006

Do 6 problems, 2 from each of the three sections. If you work on more than 6 problems, or on more than 2 from any section, clearly indicate which you want graded. Different parts of a problem do not necessarily count the same.
Justify everything carefully. You may quote and use well-known theorems, provided they do not make the problem trivial. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem or appeal to known results in such a way that the problem becomes trivial.
You should have no need for a calculator on this exam, but you may, if you wish, use your calculator for routine computations with numbers. You may not use any algebra software that might be installed on your calculator. Cell phones are not allowed.
Note: $\mathbb{Q}$ denotes the field of rational numbers. The ring of integers is denoted by $\mathbb{Z}$.

Section I: Groups and Geometry

Because of the evolving syllabus in Math 817-818, problems in this section are designated "**S**" or "**NS**". An "**S**" indicates that the problem is likely to involve the Sylow theorems while an "**NS**" indicates that the Sylow theorems are not needed to solve the problem. You may choose any two of the four problems from this section.

1. (**NS**) Let A and B be subgroups of a group D, where $|A| = 6$ and $|B| = 9$.

(a) If either A or B is normal, determine the possible orders of the subgroup C of D generated by A and B. For each possibility, give an explicit example showing that the possibility actually occurs. Be sure to prove that no other orders can occur.

(b) Give an explicit example showing that more possibilities occur, if neither A nor B is required to be normal.

2. (**NS**) Let G be a group of order p^n, where p is prime and $n \geq 1$.

(a) Prove that the center $\mathrm{Z}(G)$ is non-trivial.

(b) Prove that G has a subgroup of order p^{n-1}.

3. (**NS**) Let H be a proper subgroup of the group O_2 of 2×2 orthogonal real matrices. Show that H is normal if and only if every element of H has determinant 1.

4. (**S**) Let G be a group of order $255 = 3 \cdot 5 \cdot 17$.

(a) Show G contains a subgroup of order 15.

(b) Prove that G is abelian.

Section II: Linear Algebra

5. Let A and B be 2×2 matrices with entries in a field k.

(a) Show that some power of A is the 0 matrix (i.e., A is *nilpotent*) if and only if there is a basis v_1, v_2 of k^2 such that $Av_1 \in kv_2$ and $Av_2 = 0$.

(b) Prove or give an explicit counterexample: if A and B are nilpotent, then so is AB.

6. Let $F : V \to V$ be the linear operator given by d/dx, where V is the subspace (of the real vector space of differentiable functions) spanned by e^x, xe^x and x^2e^x. Find the Jordan Canonical Form A for F and a basis of V with respect to which the matrix for F is A.

7. Let k be a field. For any subspace W of k^n, define W' to be the subspace of all vectors $(b_1, \ldots, b_n) \in k^n$ such that $a_1b_1 + \cdots + a_nb_n = 0$ for all $(a_1, \ldots, a_n) \in W$.

(a) Show that for any subspace W, we have $\dim W + \dim W' = n$.

(b) Show that for any subspace W, we have $(W')' = W$.

(c) Show by example that $W + W'$ need not equal k^n, but show that $W + W' = k^n$ holds for all W provided k is the field of real numbers.

Section III: Rings and Fields

8. Let k be a field, and let $p = (a, b) \in k^2$ be a point in the k-plane. Let $I_p \subset k[x, y]$ be the ideal in the polynomial ring $k[x, y]$ generated by $x - a$ and $y - b$. Likewise, let I_q be the ideal for a distinct point $q = (c, d)$.

(a) For each $f \in k[x, y]$, show that there exist $g \in I_p$ and $h \in I_q$ such that $f = g + h$.

(b) For each pair of elements $i, j \in k$, show that there is a polynomial $f_1 \in k[x, y]$ such that $f_1(p) = i$ and $f_1(q) = j$, and that if $f_2 \in k[x, y]$ also has $f_2(p) = i$ and $f_2(q) = j$, then $f_1 - f_2 = g_1h_1 + \cdots + g_mh_m$ for some $g_1, \ldots, g_m \in I_p$ and some $h_1, \ldots, h_m \in I_q$.

9. Prove that $x^4 + 2x^3 + x^2 + 4x + 3$ is irreducible over the rationals.

10. Find a prime factorization of $54 - 18i$ in the Gaussian integers $\mathbb{Z}[i]$. Explain how you obtain your answer.