Math 817–818 Qualifying Exam

January 17, 2007, 2–6 pm

Solve two problems from each of the three parts, for a total of *six* problems. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. **Justify all your answers**.

Section I: Groups

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(1) Let G be a group of order 105. Prove that G contains a cyclic normal subgroup of order 35.

(2) Consider the following statement, which can be viewed as a converse to Lagrange's theorem: *Let G be a finite group of order n and d a positive divisor of n. Then G has a subgroup of order d.*

   (a) Give a counterexample (with complete justification) to the above statement.

   (b) Prove that if one adds the hypothesis that G is abelian, then the above statement is true.

(3) Let n be a positive integer and set

$$d(n) := \sup\{\, d \mid \text{ there exists an abelian subgroup } H \text{ of } S_n \text{ of order } d\}.$$

Let $\Lambda(n)$ denote the set of all subgroups of S_n of order $d(n)$. Prove or disprove: For all $n \geq 1$, $\Lambda(n)$ contains a cyclic group.

Section II: Linear Algebra

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(4) Let $\Lambda = \{A \in GL_3(\mathbb{C}) \mid A^3 = I\}$. Find (with justification) the cardinality of the set $\{\text{trace}\,(A) \mid A \in \Lambda\}$.

(5) Let A and B be $n \times n$ complex matrices such that $(AB)^n = 0$. Prove that $(BA)^n = 0$. (Hint: First prove that if C is a $n \times n$ matrix such that $C^k = 0$ for some k then $C^n = 0$.)

(6) Prove the Spectral Theorem: Every normal matrix over $\mathbb{C}$ is unitarily similar to a diagonal matrix. [For your convenience, we recall some standard definitions, for square matrices over $\mathbb{C}$. The *conjugate transpose* A^* of a matrix $A = [a_{ij}]$ is the matrix whose ij-entry is $\overline{a_{ji}}$. A square matrix P is *unitary* provided $PP^* = I$, the identity matrix. A square matrix P is *normal* provided $PP^* = P^*P$. Two square matrices A and B are *unitarily similar* provided there is a unitary matrix P such that $P^*AP = B$.]

Section III: Rings and Fields

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(7) Consider the ideal I in the polynomial ring $\mathbb{R}[x]$ generated by elements $x^3 - x^2 + x - 1$ and $x^3 + x^2 + x + 1$.

(a) Prove or disprove: The ideal I is principal. If I is principal, find a generator for I.

(b) Is the ideal I prime?

(c) Is the ideal I maximal?

(8) Find all prime ideals in the ring $\mathbb{Z}[x]/(12, x^2 + 1)$.

(9) For each positive integer n let $R_n = \mathbb{Z}[2^{\frac{1}{2}}, 3^{\frac{1}{3}}, \ldots, n^{\frac{1}{n}}]$. This is a subring of $\mathbb{R}$, the field of real numbers.

(a) Prove that R_n is Noetherian for all $n \geq 1$.

(b) Let $R = \bigcup_{n=1}^{\infty} R_n$. Prove that R is not Noetherian.

(10) Set $\xi = e^{2\pi i/5}$. Find the Galois group of $\mathbb{Q}(\xi)$ over $\mathbb{Q}$. (Of course, you should completely justify your answer.)