Math 817–818 Qualifying Exam

January 2008

Rules of the game:

(a) Solve *two* problems from each of the three parts, for a total of *six*.

For problems with multiple parts you can assume the results of earlier parts, even if you have not solved them.

If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(b) **Justify all of your answers**.

(c) Bold numbers in [**brackets**] indicate the number of points assigned for a complete solution.

Section I: Groups

(1) [**20**] Prove that no group of order 224 ($=2^5 7$) is simple.

(2) Let $F = \mathbb{F}_4$, the field with 4 elements, and let G denote the group $\mathrm{SL}_2(F)$ of 2×2 matrices with elements in F and determinant equal to 1.

(a) [**5**] Prove that the order of G is 60.

(b) [**5**] Prove that the natural linear G-action on F^2 induces a G-action on the set $\mathbb{P} = (F^2 \setminus 0)/\sim$, where $\sim$ is the equivalence relation:

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{if} \quad \begin{bmatrix} x \\ y \end{bmatrix} = c \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{for some } c \in F^\times .$$

(c) [**10**] Using the action in (b) prove that G is isomorphic to A_5, the alternating group on 5 letters.

[Hint: You can use the conclusion of Problem (4.a) below. If it helps, you may use without proof the fact that A_5 is simple.]

(3) Let P be a Sylow p-subgroup of G.

(a) [**8**] Prove that P is the only Sylow p-subgroup in $N_G(P)$.

(b) [**12**] Prove that $N_G(P) = N_G(N_G(P))$.

Section II: Linear Algebra

(4) Let V be a finite dimensional vectorspace over a field F.

(a) [**10**] Let $f\colon V \to V$ be a linear transformation, such that each non-zero vector in V is an eigenvector for f. Prove that there is a scalar $c \in F$, such that $f = c \cdot \mathrm{id}_V$ for some $c \in F$.

(b) **[10]** Prove that when F is infinite V is not a union of finitely many proper subspaces.

(5) Let V be a finite dimensional vector space over a field and $f\colon V \to V$ a linear map.

(a) **[5]** Prove that there exists an integer i such that is $\mathrm{Ker}(f^n) = \mathrm{Ker}(f^{n+1})$ holds for all $n \geq i$.

(b) **[8]** Let s denote the least integer i with the property described in (a). Prove that $\mathrm{Ker}(f^{n-1}) \neq \mathrm{Ker}(f^n)$ holds for all $n \leq s$.

(c) **[7]** Prove that $s = \inf\{n \geq 0 \,|\, \mathrm{Im}(f^n) = \mathrm{Im}(f^{n+1})\}$.

(6) Let $A = \begin{bmatrix} 6 & 4 & 2 \\ 2 & 7 & 4 \end{bmatrix}$, regarded as a matrix over $\mathbb{Z}$.

(a) **[10]** Find invertible integer matrices P and Q such that the matrix $B = P^{-1}AQ$ has $b_{ij} = 0$ for all $i \neq j$.

(b) **[10]** Find all the integer solutions of the matrix equation $AX = 0$, where X is a column vector of indeterminates.

Section III: Rings and Fields

(7) Let $a_1, \dots a_n$ be rational numbers, and assume $F = \mathbb{Q}(\sqrt{a_1}, \dots, \sqrt{a_n})$ has degree 2^n over $\mathbb{Q}$.

(a) **[5]** Prove that F is the splitting field of the polynomial $\prod_{i=1}^{n}(x^2 - a_i)$.

(b) **[7]** Prove that $\sqrt{a_1}, \dots, \sqrt{a_n}$ are linearly independent over $\mathbb{Q}$.

(c) **[8]** Prove that $\sqrt{a_1} + \cdots + \sqrt{a_n}$ is a primitive element for F over Q.

(8) Consider the extension of fields $\mathbb{Q} \subset \mathbb{Q}(\zeta)$, where ζ is the complex number $\cos(2\pi/13) + i\sin(2\pi/13)$.

(a) **[5]** Determine the Galois group of this extension.

(b) **[7]** List the distinct subgroups of the Galois group.

(c) **[8]** Find a primitive element for each subfield of $\mathbb{Q}(\zeta)$.

(9) Consider the ring $R = \mathbb{R}[x]/(x^2 + 1)^2$.

(a) **[5]** Prove that the ring $R/(x^2 + 1)$ is isomorphic to $\mathbb{C}$.

(b) **[5]** Find in R an element r satisfying $r^2 + 1 = 0$.
[Hint: What is the image of r in $\mathbb{C}$?]

(c) **[10]** Prove that one has an isomorphism $R \cong \mathbb{C}[y]/(y^2)$ of rings.