Math 817–818 Qualifying Exam

January 2012

Instructions:

(a) Solve *two* problems from each of the three parts, for a total of *six*.

For problems with multiple parts you can assume the results of earlier parts, even if you have not worked on them.

If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(b) **Justify all of your answers**.

(c) Each problem is worth 20 points. For problems with multiple parts, bold numbers in [**brackets**] indicate the number of points assigned for that part.

Section I: Groups

(1) Let G be a (not necessarily finite) group and H and K normal subgroups such that $G = HK$. Prove that $G/(H \cap K) \cong G/H \times G/K$.

(2) Suppose G is a finite group which has precisely one subgroup of order d for each divisor d of $|G|$. Prove that G is cyclic.

(3) Let G be a group of order $90 = 2 \cdot 3^2 \cdot 5$ and let $\mathrm{Syl}_3(G)$ denote the set of Sylow 3-subgroups of G.

(a) [**10**] Suppose for any $Q, Q' \in \mathrm{Syl}_3(G)$ either $Q = Q'$ or $Q \cap Q' = \{1\}$. Prove that G is not simple.

(b) [**10**] Suppose there exists $Q, Q' \in \mathrm{Syl}_3(G)$ such that $|Q \cap Q'| = 3$. Prove that G is not simple. (Hint: Consider the normalizer of $Q \cap Q'$.)

Section II: Rings and Fields

(4) Let F be a field of characteristic $p > 0$, $a \in F$, and consider the polynomial $f(x) = x^p - x - a \in F[x]$.

(a) [**10**] Prove that $f(x)$ is either irreducible over F or splits into linear factors over F. (Hint: If α is a root of $f(x)$ in some splitting field, consider $\alpha + i$ for $i \in \mathbb{F}_p$, the field of p elements.)

(b) [**10**] Suppose $f(x)$ is irreducible over F. Prove that the Galois group of $f(x)$ over F is cyclic.

(5) Let $\alpha = \sqrt{2} + \sqrt{5} \in \mathbb{R}$.

(a) **[6]** Find the minimal polynomial $f(x)$ of α over $\mathbb{Q}$.

(b) **[7]** Let E be the splitting field of $f(x)$ over $\mathbb{Q}$. Find the Galois group G of $E/\mathbb{Q}$.

(c) **[7]** Find all subgroups of G and generators for the corresponding intermediate fields of $E/\mathbb{Q}$.

(6) Let F be any field, and let x, y, and t be indeterminates. Prove in detail that $F[x, y]/(y^2 - x^3)$ and $F[t^2, t^3]$ are isomorphic rings.

Section III: Linear Algebra and Modules

(7) Consider the following matrix over the complex numbers:

$$A = \begin{bmatrix} -1 & -9 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 2 & 7 & 2 & 0 \\ 4 & 13 & 0 & 2 \end{bmatrix}.$$

(a) **[5]**Show that the characteristic polynomial of A is $(t-2)^4$.

(b) **[6]** Find the Jordan canonical form J of A.

(c) **[9]** Find an invertible matrix P such that $P^{-1}AP = J$.

(8) Let N be a submodule of an R-module M. Using Zorn's Lemma, prove that there is a submodule N' such that

(i) $N \cap N' = (0)$, and

(ii) $N'' \cap (N + N') \neq (0)$ for every non-zero submodule N'' of M.

(9) Let A be a square matrix over the field $\mathbb{C}$ of complex numbers.

(a) **[13]** Suppose A is invertible. Prove that there is a square matrix B over $\mathbb{C}$ such that $B^2 = A$. (Hint: Reduce to the case that $A = I + N$ where I is the identity matrix and N is a nilpotent matrix.)

(b) **[7]** Show by example that (a) can fail if A is not assumed to be invertible.