Math 817–818 Qualifying Exam

January 2014

Instructions:

(a) Solve *two* problems from each of the three parts, for a total of *six*.

For problems with multiple parts you can assume the results of earlier parts, even if you have not solved them.

If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(b) **Justify all of your answers.**

(c) Bold numbers in **[brackets]** indicate the number of points assigned for a complete solution.

Section I: Group Theory

(1) Prove that any group of order 36 has a normal subgroup of order 3 or 9.

(2) Let G be a finite group and H a subgroup of G of index h.

(a) **[12]** Prove that H has at most h conjugates in G.

(b) **[8]** Prove that if $H \neq G$ then $\bigcup_{g \in G} gHg^{-1} \neq G$.

(3) Let p be an odd prime, G a group of order $p(p+1)$, and P a Sylow p-subgroup of G. Assume that P is **not normal** in G.

(a) **[6]** Prove that the normalizer of P in G is exactly P.

(b) **[7]** Let $P = \langle t \rangle$ and a an element of order 2 in G (such an element exists because $p+1$ is even). Prove the elements

$$a,\ tat^{-1},\ t^2at^{-2}, \ldots\ t^{p-1}at^{-(p-1)}$$

are all distinct.

(c) **[7]** Prove that $p+1$ is a power of 2. (Hint: Count elements.)

Section II: Field Theory

(4) Let F be a field and F^* its group of units.

(a) **[10]** Prove that any finite subgroup of F^* is cyclic.

(b) **[10]** Suppose F is algebraically closed and has characteristic $p > 0$. For any positive integer n, prove that F^* has a subgroup of order n if and only if p does not divide n.

(5) Let E be a subfield of $\mathbb{C}$ and suppose every element of E is a root of a polynomial of degree 10 in $\mathbb{Q}[x]$. Prove that $[E : \mathbb{Q}] \leq 10$. (Note: E is not assumed to be a finite extension of $\mathbb{Q}$.)

(6) Let $\omega \in \mathbb{C}$ be a primitive 25th root of unity.

(a) **[10]** Find $[Q(\omega) : \mathbb{Q}]$ and generator(s) for $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$.

(b) **[10]** Draw the subfield lattice for $\mathbb{Q}(\omega)$ and indicate the degrees of each extension. (You do not have to find field generators for each of the subfields.)

Section III: Rings, Modules, and Linear Algebra

(7) Prove that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. Justify all details.

(8) Let R be a commutative ring with identity and M an R-module. Let $f : M \to M$ be a surjective R-module homomorphism.

(a) **[7]** Give an example of such an R, M, and f such that f is not an isomorphism.

(b) **[13]** Prove that if M is Noetherian then f is an isomorphism.

(9) Let F be a field and A a square matrix with entries from F. Prove that A is similar to its transpose.