Math 817–818 Qualifying Exam

January 2015

Instructions:

(a) Solve *two* problems from each of the three parts, for a total of *six*.

The problems have multiple parts; usually each part is a step towards the next. Assume the results of earlier parts, even if you do not solve them.

If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(b) **Justify all of your answers**.

(c) Bold numbers in [**brackets**] indicate the number of points assigned for a complete solution.

Section I: Group Theory

(1) Let G be a finite group and let H be a subgroup of G of index p, where p is the smallest prime divisor of the order of G. Prove that H is normal in G.

(2) Let G be a group of order $385 = 5 \cdot 7 \cdot 11$. Prove that G has a normal subgroup of order 11 and that the center of G contains a subgroup of order 7.

(3) Let G be a group and H a subgroup of G. Recall that the centralizer of H in G is

$$C_H(G) = \{g \in G \,|\, gh = hg \text{ for all } h \in H\}.$$

Prove that if H is normal in G, then so is $C_G(H)$ and that $G/C_G(H)$ is isomorphic to a subgroup of the automorphism group of H.

Section II: Field Theory and Galois theory

(4) Let $L \subseteq F \subseteq E$ be fields.

(a) [**12**] Prove that E is algebraic over L if and only if E is algebraic over F and F is algebraic over L.

(b) [**8**] Give an example where E/F and F/L are finite Galois extensions but E/L is not Galois.

(5) Let E/F be a finite Galois extension and let G be the Galois group of E/F. Suppose that $E = F(\alpha)$ and let $f(x)$ be the minimal polynomial of α over F. Prove that

$$f(x) = \prod_{\sigma \in G} (x - \sigma(\alpha)).$$

(6) Let F be a field and G a finite subgroup of the multiplicative group F^*. Prove that G is cyclic.

Section III: Ring theory and Linear Algebra

(7) Let F be a field and A a square matrix with entries from F. Prove that A is similar to its transpose.

(8) Consider the following matrix:

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}.$$

(a) **[10]** Find the rational canonical form of A.

(b) **[10]** Find the Jordan canonical form of A.

(9) Let R be a commutative ring with identity (with $1 \neq 0$) and I a proper ideal. Prove there exists a prime ideal p containing I such that whenever $p \supseteq q \supseteq I$ where q is also a prime ideal, then $p = q$. (Hint: use Zorn's lemma.)