Ph.D. Qualifying Exam and Masters Comprehensive Exam
Algebra: 817–818. June 1, 1998.

Do exactly **two** problems from each section (for a total of **six** problems). If you work on more than six problems, or on more than two from any one section, clearly indicate which you want graded. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

A clean TI-85 is being provided for you to use for routine calculations if you want, but be sure to provide written justifications of all answers.

Section I: Groups

For a group G, let $Z(G)$ denote the center of G.

1. Let G be a group and set $T = G \times G$. Define
$$D = \{(g, g) \in G \times G \mid g \in G\}$$
   (a) Prove that D is a subgroup of T which is isomorphic to G.
   (b) Prove that D is a normal subgroup of T if and only if G is abelian.

2. Let G be a finite group and let H be a subgroup of G of index n. Suppose that $n! < |G|$. Prove that there is a normal subgroup K of G with $1 \neq K \leq H$.

3. Let G be a group of order $385 = 5 \cdot 7 \cdot 11$. Prove that G contains a normal subgroup of order 11 and that $Z(G)$ contains a subgroup of order 7.

4. Let p be a prime and n a positive integer. Let G be a group of order p^n.
   (a) Prove that $Z(G) \neq \{1_G\}$.
   (b) Suppose G is a nonabelian group of order p^3. Prove that $G/Z(G) \cong Z_p \times Z_p$, where Z_p is the cyclic group of order p.

Section II: Rings and Fields

5. Let α be a complex number such that $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is odd. Show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$.

6. Let R be an integral domain, F a field, and $\phi : R \to F$ a nontrivial ring homomorphism (so $\phi(1_R) = 1_F$). Set $D = R \setminus \operatorname{Ker} \phi$.
   (a) Show that D is multiplicatively closed, $1_R \in D$, and that $0 \notin D$.
   (b) Show that there is a ring homomorphism $\Phi : D^{-1}R \to F$ such that $\phi = \Phi \circ \iota$, where $\iota : R \to D^{-1}R$ is the map $\iota(r) = r/1_R$. (Note: Recall that elements of $D^{-1}R$ are equivalence classes, so you'll have to show that your Φ is well-defined.)

7. Let R be a commutative ring with identity and I an ideal of R. The *radical* of I is defined to be
$$\mathcal{R}(I) = \{r \in R \mid r^n \in I \text{ for some integer } n > 0\}$$
   (a) Prove that $\mathcal{R}(I)$ is an ideal of R containing I.
   (b) Suppose that P is a prime ideal of R. Prove that $\mathcal{R}(P) = P$.

8. Prove that the polynomial $x^4 + 4x^3 + 11x^2 + 10x + 5$ is irreducible over $\mathbb{Q}$. Hint: Go mod 3.

Section III: Linear Algebra

9. Let R be a ring with 1 and M a left R-module. Suppose $\phi : M \to M$ is a homomorphism satisfying $\phi \circ \phi = \phi$. Prove that $M \cong \operatorname{Ker} \phi \oplus \operatorname{Im} \phi$.

10. Let F be a field and V a vector space (not necessarily finite-dimensional) over F. Let S be a set of linearly independent vectors of V. Prove that there is a basis for V containing S.

11. Determine all possible Jordan canonical forms for a 6×6 matrix with minimal polynomial $(x + 1)(x - 3)^3 = x^4 - 8x^3 + 18x^2 - 27$. For each, also give the corresponding list of invariant factors and the corresponding list of elementary divisors.

12. Let
$$A = \begin{pmatrix} 8 & -6 & -18 \\ 9 & -7 & -27 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ -21 & 6 & -1 \\ -48 & 16 & -2 \end{pmatrix}$$
Are A and B similar matrices? Why or why not?