Ph.D. Qualifying Exam
Algebra: 901-902. June 1, 1999

Do a total of four problems, one from each section. In sections III and IV, all rings are assumed to be commutative with identity.

Section I: Groups

1. Classify up to isomorphism all groups of order 28.

2. Prove that every group of order $5 \cdot 19 \cdot 29$ is cyclic.

Section II: Fields

3. Let E/K be an algebraic extension and $\sigma : E \to E$ a field homomorphism fixing K. Prove that σ is an isomorphism.

4. Let E be the splitting field of $x^6 + x^3 + 1$ over $\mathbb{Q}$.

(a) Find the Galois group of $E/\mathbb{Q}$.
(b) Find field generators for all of the intermediate fields of $E/\mathbb{Q}$.

5. Let E/K be a separable algebraic extension such that every nonconstant polynomial in $K[x]$ has a root in E. Prove that E is an algebraic closure of K. (Hint: use the primitive element theorem.)

Section III: General Rings and Modules

6. Let I be an ideal of R and M a finitely generated R-module such that $IM = M$. Prove that there exists $s \in I$ such that $(1 - s)M = 0$.

7. Let R be a ring with a unique maximal ideal. Prove that every finitely generated projective R-module is free.

8. Let (#) denote the exact sequence of R-modules $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$.

(a) Prove that (#) splits if and only if the natural map $\mathrm{Hom}_R(N, M) \xrightarrow{g^*} \mathrm{Hom}_R(N, N)$ is surjective.
(b) Suppose N is finitely presented. Prove that (#) splits if and only if for all $p \in \operatorname{Spec} R$, $0 \to L_p \to M_p \to N_p \to 0$ splits.

Section IV: Noetherian rings and modules

9. Let R be a Dedekind domain and M a finitely generated R-module. Prove that M is projective if and only if it is torsion-free.

10. Let $R \subseteq S$ be Noetherian rings such that S is integral over R. Suppose R is local. Prove that S has only finitely many maximal ideals.

11. Let R be a Noetherian ring and I and J two ideals of R. Prove that $I = J$ if and only if $I_p = J_p$ for all $p \in \mathrm{Ass}_R(R/I) \bigcup \mathrm{Ass}_R(R/J)$.