Master's Exam and Ph.D. Qualifying Exam
Algebra: Math 817-818, June 5, 2001

Do 6 problems, 3 from each of the two sections. If you work on more than six problems, or on more than three from any section, then clearly indicate which ones you want graded. For problems with more than one part, do not assume that all parts count equally. There are 10 problems on two pages.

If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Note: $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers respectively, and $\mathbb{Z}$ denotes the ring of integers. The symbol i is only used to denote the usual element of $\mathbb{C}$.

Section I: Rings and Fields

1. Let R be a unique factorization domain and d a nonzero element of R. Prove that there are only a finite number of distinct principal ideals that contain the ideal (d).

2. (a) Show that in $\mathbb{C}$, $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ are isomorphic as vector spaces, but not as fields.
(b) Show that any quadratic extension $\mathbb{R}(\alpha)$ of $\mathbb{R}$ is isomorphic to $\mathbb{C}$ (as a field).

3. (a) Determine a presentation matrix for the ideal $(13, -5+12i, 7-9i)$ of $\mathbb{Z}[i]$, viewed as a $\mathbb{Z}[i]$-module.
(b) Use the matrix to determine if the ideal is a cyclic $\mathbb{Z}[i]$-module.

4. (a) Prove that $f(x) = x^7 - \frac{35}{3}x^6 + 42x^3 + 7x - \frac{14}{5}$ is irreducible in $\mathbb{Q}[x]$.
(b) Prove that $f(x)$ is irreducible in $\mathbb{Q}(i)[x]$. (Hint: Let $\alpha \in \mathbb{C}$ be a root of $f(x)$, and think about degrees of various field extensions.)

5. (a) Construct a field F with 27 elements, making sure to give explicit rules for addition and multiplication in F.
(b) How many fields does F contain? Give brief justification.

Section II: Groups and Linear Algebra

6. If $f : G \longrightarrow H$ is a homomorphism, H is abelian, and N is a subgroup of G containing $\ker f$, then show that N is normal in G.

7. Let G be a group with $|G| = p^n$, where p is prime and $n \geq 2$.
   (a) Recall that the *center of* G is the set $Z(G)$ of elements which commute with all elements of the group:
   $$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}.$$
   Show that $|Z(G)| \geq p$.
   (b) Recall that for $x \in G$, the *centralizer of* x is the set $Z(x)$ of elements which commute with x:
   $$Z(x) = \{g \in G \mid gx = xg\}.$$
   Show that $|Z(x)| \geq p^2$.
   (c) Show that if $|G| = p^2$, then either G is isomorphic to the cyclic group C_{p^2} or G is isomorphic to the product of cyclic groups $C_p \times C_p$.

8. Let G be the group of symmetries of an equilateral triangular lattice $L \subset \mathbb{R}^2$. Find $[G : T \cap G]$, where T is the group of translations of $\mathbb{R}^2$, and justify your answer.

9. (a) Find a unitary matrix P so that PAP^* is diagonal, if
   $$A = \begin{bmatrix} 3 & 2i \\ -2i & 3 \end{bmatrix}.$$
   (b) Find a real orthogonal matrix Q so that QBQ^t is diagonal, if
   $$B = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 9 & -3 \\ 0 & -3 & 1 \end{bmatrix}.$$

10. (a) Prove that a Jordan block has a one-dimensional space of eigenvectors.
    (b) Prove that, conversely, if the eigenvectors of a complex matrix A are multiples of a single vector, then the Jordan form for A consists of one block.