Math 817–818 Qualifying Exams and Masters Comprehensive Exam

June 6, 2003 1:00–4:00PM

- Do *two* of the four given problems from each of the three sections, for a total of *six* problems. Be sure to make it clear which six problems you want graded.
- If you have doubts about the wording of a problem or which results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.
- Be sure to show your reasoning clearly and explain everything carefully.

I Groups and Geometry

I.1 Recall that the 3×3 orthogonal group, O_3, is defined as

$$O_3 = \{A \in GL_3(\mathbb{R}) \mid A^T A = I_3\}.$$

Prove O_3 is an internal direct product of two non-trivial subgroups, H and K (i.e., prove that for some pair of non-trivial subgroups H and K of G, there is an isomorphism $H \times K \cong G$ given by $(h, k) \mapsto hk$).

I.2 Let $\mathbf{M}$ denote the group of rigid motions of the plane and let G be a finite subgroup of $\mathbf{M}$. Prove G fixes a point — that is, show there exists a point P in the plane such that $g(P) = P$ for all P. (You are *not* allowed to use the theorem that classifies all finite subgroups of $\mathbf{M}$.)

I.3 Let G be a non-zero subgroup of the group of real numbers under addition $(\mathbb{R}, +)$. Prove G is discrete if and only if G is an infinite cyclic group. (Recall that G is *discrete* if there is a real number $\epsilon > 0$ such that for all non-zero elements $0 \neq g \in G$, we have $|g| > \epsilon$. Here, $|g|$ denotes the absolute value of g, not its order.)

I.4 Let N be a normal subgroup of a finite group G. Assume $|N| = p$, where p is prime, and that p is the smallest prime divisor of $|G|$. Prove N is contained in the center of G. *Hint:* Consider the action of G on N via conjugation.

(Turn the page for sections II and III.)

II Linear Algebra

II.1 An $n \times n$ matrix A is called *unipotent* if $A = I_n + B$ for some nilpotent matrix B. (A matrix B is nilpotent if $B^k = 0$ for some $k \geq 1$.) Prove that if A is unipotent, then A is similar to a lower triangular matrix with 1's along the diagonal:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ * & \cdots & * & 1 & 0 \\ * & \cdots & * & * & 1 \end{bmatrix}$$

II.2 Let A be an $n \times n$ matrix with complex entries.

(a) Prove that if $A^k = I_n$ for some $k \geq 1$, then A is diagonalizable.

(b) Show by example that even if the matrix A in part (a) has real entries, it need *not* be diagonalizable over the reals.

II.3 Find the rational canonical form of the matrix with complex entries

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ z & 1 & 0 & 1 \end{bmatrix},$$

where z is an arbitrary complex number. (Your answer may or may not depend on the value of z.)

I.4 Recall that, if A is an $n \times n$ complex matrix, then A is *Hermitian* if $A = A^*$ and A is *unitary* if $A^*A = I_n$. (Here, A^* denotes the conjugate transpose of A.) Assume A is Hermitian. Prove that A is unitary if and only if all the eigenvalues of A are ± 1.

III Rings, Modules, and Fields

III.1 Let $f(x), g(x) \in \mathbb{Q}[x]$ be irreducible polynomials, and let $\alpha \in \mathbb{C}$ be a root of $f(x)$ and let $\beta \in \mathbb{C}$ be a root of $g(x)$. Prove that $f(x)$ is irreducible over $\mathbb{Q}(\beta)$ if and only if $g(x)$ is irreducible over $\mathbb{Q}(\alpha)$.

III.2 Let $R = \mathbb{Z}[\sqrt{-10}]$.

(a) Using that $(2 + \sqrt{-10})(2 - \sqrt{-10}) = 14$, prove R is not a UFD.

(b) Prove $I = (7, 2 + \sqrt{-10})$ is a maximal ideal of R that is *not* principal.

III.3 Prove $f(x) = 25x^5 - 6x^4 - x^2 + 5x - 16 \in \mathbb{Q}[x]$ is irreducible. *Hint:* Modulo 3, we have $f(x) = (x^2 + 1)(x^3 - x - 1)$.

III.4 If M is a $\mathbb{Z}$-module (i.e., an abelian group), the *annihilator* of M is defined to be

$$ann(M) = \{n \in \mathbb{Z} \mid nx = 0, \text{for all } x \in M\}.$$

In general, $ann(M)$ is an ideal of $\mathbb{Z}$. (You need not prove this.)

Suppose M is the $\mathbb{Z}$-module presented by an $n \times n$ matrix A with entries in $\mathbb{Z}$ (so that $M \cong coker(A)$).

(a) Prove $\det(A) \in ann(M)$.

(b) Assume $\det(A) \neq 0$. Prove $ann(M) = (det(A))$ if and only if M is cyclic. (Here, $(det(A))$ denotes the principal ideal of $\mathbb{Z}$ generated by $det(A)$.)