Math 817–818 Qualifying Exam and Masters Comprehensive Exam

June 1, 2004 1:00–4:00PM

- Do *two* of the four given problems from each of the three sections, for a total of *six* problems. Be sure to make it clear which six problems you want graded.
- If you have doubts about the wording of a problem or which results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.
- Be sure to show your reasoning clearly and explain everything carefully.

I. Linear Algebra

1. Let A be a 7×7 matrix with rational entries.
   (a) If the characteristic polynomial of A is $(t-2)^7$, the minimal polynomial of A is $(t-2)^3$, and the dimension of the eigenspace of A corresponding to eigenvalue 2 is 3, how many possible similarity classes for A are there? Explain.
   (b) The matrix
$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$
   satisfies all the properties of part (a). Give the rational canonical form and the Jordan canonical form for A. *Hint:* One way of doing this is to note that
$$(A-2I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$
2. Let F be a field and $\phi : F^n \to F^m$ a linear transformation given by left-multiplication by the $m \times n$ matrix A. Prove that the following conditions are equivalent:
   (a) A has a right inverse, i.e., there is an $n \times m$ matrix B such that $AB = I_m$, the $m \times m$ identity matrix.
   (b) ϕ is surjective.
   (c) A has rank m.
3. Let A and B be $n \times n$ complex Hermitian matrices. Prove that A and B are similar if and only if they have the same eigenvalues, counted with multiplicity.
4. Let A be an $n \times n$ real matrix, all of whose eigenvalues are real numbers. Prove that there is an orthogonal $n \times n$ matrix P such that PAP^t is upper triangular.

*Turn the page over for parts **II** and **III**.*

II. Groups and Geometry

5. Let G be the subgroup of $GL_2(\mathbb{C})$ consisting of upper triangular matrices whose diagonal entries are the same, i.e., all invertible complex matrices of the form

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

Prove $G \cong \mathbb{C}^\times \times \mathbb{C}$ (where $\mathbb{C}^\times$ is the group of non-zero complex numbers under multiplication and $\mathbb{C}$ is the group of all complex numbers under addition).

6. Let G be a nonabelian group of order 21.
   (a) Prove that there is only one possible class equation for G.
   (b) How many elements of each order must G have?

7. Recall that a subgroup H of M (the group of rigid motions of the plane) is *discrete* if there is an $\epsilon > 0$ such that
   - if ρ is a nontrivial rotation in H of angle α, then $|\alpha| \geq \epsilon$
   - if t is a nontrivial translation in H by the vector v, then $\|v\| \geq \epsilon$.

   If $\mathcal{O}$ is the subgroup of M fixing the origin and H is a discrete subgroup of $\mathcal{O}$, prove that H is finite.

8. Let G be a group and H and K subgroups of G. For $a \in G$, define the H-K *double coset of* a to be

$$HaK = \{hak \mid h \in H, k \in K\}.$$

   (a) Prove that the H-K double-cosets partition G.
   (b) Prove that, in general, the H-K double-cosets do not all have the same number of elements. In other words, give an explicit example of a finite group G, subgroups H and K of G, and elements $a, b \in G$ such that $|HaK| \neq |HbK|$.
   Hint: Consider a non-normal subgroup H of G.

III. Rings, Fields, and Modules

9. Let $F = \mathbb{F}_{2187}$ be the field with $2187 = 3^7$ elements.
   (a) Prove that $F = \mathbb{F}_3(\alpha)$ for every $\alpha \in F \setminus \{0, 1, 2\}$.
   (b) How many irreducible polynomials of degree 7 are there over $\mathbb{F}_3$? Justify your answer.

10. Let R be a commutative ring and let I be an ideal of R.
   (a) Prove or disprove: If R/I is a free R-module then $I = 0$.
   (b) Assume $I \neq \{0\}$. Prove that I is a free R-module if and only if $I = (\alpha)$, where $\alpha \in R \setminus \{0\}$ is not a zero divisor.

11. Let $R = \mathbb{Z}[\sqrt{2}]$. Prove that $R/(1 + 3\sqrt{2}) \cong \mathbb{Z}/17\mathbb{Z}$ (as rings).

12. Let $f(t, x) \in \mathbb{C}[t, x] \cong (\mathbb{C}[t])[x]$ be written as a polynomial in x whose coefficients are polynomials in t:

$$f(t, x) = a_n(t)x^n + \cdots + a_1(t)x + a_0(t).$$

   Prove the following version of Eisenstein's Criterion: Suppose that
   - $a_n(0) \neq 0$
   - $a_{n-1}(0) = \cdots = a_0(0) = 0$
   - $a_0'(0) \neq 0$
   - If $f(t, x) = g(t)h(t, x)$, then $g(t) \in \mathbb{C}$. (In other words, $f(t, x)$ has no nontrivial factor which is a polynomial in t alone.)

   Prove that $f(t, x)$ is irreducible in $\mathbb{C}[t, x]$.