Master's Comprehensive and Ph.D. Qualifying Exam
Algebra: Math 817-818, June 3, 2005

Do 6 problems, 2 from each of the three sections. If you work on more than six problems, or on more than 2 from any section, clearly indicate which you want graded. Different parts of a problem do not necessarily count the same.

Justify everything carefully. You may quote and use well-known theorems, provided they do not make the problem trivial. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem or appeal to known results in such a way that the problem becomes trivial.

The use of calculators is not allowed on this exam. If you have a cell phone with you, please turn it off.

Note: $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real and complex numbers respectively. The ring of integers is denoted by $\mathbb{Z}$.

Section I: Groups

1. Recall that $\mathrm{GL}_2(\mathbb{R})$ is the group of invertible 2×2 matrices over the real numbers, and $\mathrm{SL}_2(\mathbb{R})$ is the subgroup consisting of matrices with determinant 1. Prove that the two matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

are conjugate elements of $\mathrm{GL}_2(\mathbb{R})$ but are *not* conjugate elements of $\mathrm{SL}_2(\mathbb{R})$.

2. Let G be a finite group of odd order, and let N be a normal subgroup of order 17. Prove that N is contained in the center of G.

3. Let $n \geq 2$. Prove that the alternating group A_n is the only subgroup of index 2 in the symmetric group S_n. (You may use, without proof, the fact that A_n is simple for $n \geq 5$.)

4. Prove that every subgroup H of the free abelian group $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^m$ for some $m \leq n$. (Of course you may *not* use the structure theorem for finitely generated abelian groups! Also, do not assume without proof that H is finitely generated.)

Section II: Rings and Fields

5. Prove that the polynomial $x^3 + x + 1$ has no roots in the field $\mathbb{F}_{625}$ of order 625.

6. Let $R = \mathbb{Q}[X, Y]/(Y^2 - X^3)$. Prove that R is not a principal ideal domain.

7. Prove that the polynomial $f = x^5 + 4x^4 + 10x^3 - 9x^2 + 9x + 2000$ is irreducible in $\mathbb{Q}[x]$.

8. Let R be a commutative integral domain. Suppose $P_1, \ldots, P_r$ is a list of distinct maximal ideals of R and $Q_1, \ldots, Q_s$ is another list of distinct maximal ideals of R. Assume that $P_1 \cap \cdots \cap P_r = Q_1 \cap \cdots \cap Q_s$. Prove that $r = s$ and, after possibly renumbering, $P_i = Q_i$ for each i.

Section III: Linear Algebra

9. Find the Jordan canonical form J (over the complex numbers $\mathbb{C}$) of the matrix

$$A := \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Also, find either P or P^{-1}, where P is an invertible matrix such that $PAP^{-1} = J$.

10. Reduce the matrix

$$\alpha := \begin{bmatrix} 3 & 1 & -4 \\ 2 & -3 & 1 \\ -4 & 6 & -2 \end{bmatrix}$$

to diagonal form over $\mathbb{Z}$, and express the cokernel of α as a direct sum of cyclic groups. (The cokernel of α is the quotient group $\mathbb{Z}^{(3)}/C$, where C is the subgroup of $\mathbb{Z}^{(3)}$ generated by the columns of α.)

11. Let A be an $n \times n$ matrix over $\mathbb{C}$, and assume that $A^4 = A$. Prove or give a counterexample: A is similar over $\mathbb{C}$ to a diagonal matrix.

12. Prove the Spectral Theorem: Every normal matrix over $\mathbb{C}$ is unitarily similar to a diagonal matrix. (You might start by showing that *every* square matrix over $\mathbb{C}$ is unitarily similar to a *triangular* matrix, although you need not use this approach.) [For your convenience, we recall some standard definitions, for square matrices over $\mathbb{C}$: The *conjugate transpose* A^* of a matrix $A = [a_{ij}]$ is the matrix whose ij entry is $\overline{a_{ji}}$. A square matrix P is *unitary* provided P is invertible and $P^{-1} = P^*$. A square matrix N is *normal* provided $NN^* = N^*N$. Two square matrices A and B are *unitarily similar* provided there is a unitary matrix P such that $P^*AP = B$.]