Master's Comprehensive and Ph.D. Qualifying Exam
Algebra: Math 817-818, May 30, 2006

Do 6 problems, 2 from each of the three sections. If you work on more than 6 problems, or on more than 2 from any section, clearly indicate which you want graded. Different parts of a problem do not necessarily count the same.
Justify everything carefully. You may quote and use well-known theorems, provided they do not make the problem trivial. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem or appeal to known results in such a way that the problem becomes trivial. You should have no need for a calculator on this exam, but you may, if you wish, use your calculator for routine computations with numbers. You may not use any algebra software that might be installed on your calculator. Cell phones are not allowed.
Note: $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers, respectively. The ring of integers is denoted by $\mathbb{Z}$.

Section I: Groups and Geometry

Because of the evolving syllabus in Math 817-818, this section has four problems to choose from. You may choose any two of the four problems from this section.

1. Prove that there are precisely two finite groups (up to isomorphism) that have exactly 3 conjugacy classes.

2. Let $G = \mathrm{GL}_2(\mathbb{F}_3)$, let N be the normal subgroup of G consisting of $\pm I_2$, and define $\mathrm{PGL}_2(\mathbb{F}_3) = \mathrm{GL}_2(\mathbb{F}_3)/N$. Prove $\mathrm{PGL}_2(\mathbb{F}_3)$ is isomorphic to S_4. (Here, $\mathbb{F}_3$ denotes the field with 3 elements, I_2 denotes the 2×2 identity matrix, and S_4 is the symmetric group on four symbols.)

3. Prove that if G is a group of order $132 = 2^2 \cdot 3 \cdot 11$, then G is not simple.

4. Let H be a normal subgroup of the group G of isometries (i.e., rigid motions) of the real plane. Show that H is either trivial or infinite.

Section II: Linear Algebra

5. Let A be a 3×3 complex singular matrix such that $A^3 = A$. Find all possible Jordan canonical forms for A (with justification).

6. Let F be a field and A an $m \times n$ matrix with entries from F. Prove that $\operatorname{rank} A \geq r$ if and only if there exists a nonzero $r \times r$ minor of A. (Recall that an $r \times r$ *minor* of A is the determinant of an $r \times r$ submatrix of A obtained by deleting $m - r$ rows and $n - r$ columns. By convention, any 0×0 minor is defined to be 1 and any $r \times r$ minor of A is zero when $r > \min\{m, n\}$.)

7. Let A and B be $n \times n$ complex Hermitian matrices. Prove that $\operatorname{trace}(AB) \in \mathbb{R}$.

Section III: Rings and Fields

Because of the evolving syllabus in Math 817-818, this section has four problems to choose from. You may choose any two of the four problems from this section.

9. Let R be a commutative ring with identity and let m and n be two distinct maximal ideals of R. Prove that $m \cap n = m \cdot n$. (Recall that if I and J are ideals of R, then $I \cdot J$ is, by definition, the smallest ideal containing all elements of the form ab for $a \in I$ and $b \in J$.)

11. Let p be a prime and let $\mathbb{F}_p$ denote the field with p elements. Find (with justification) the number of monic irreducible polynomials of degree 7 in $\mathbb{F}_p[x]$.

10. Let R be a commutative ring with identity. Prove that if R is Noetherian, so is $R[x]$.

12. Let $F = \mathbb{Q}(\omega)$ where ω is a primitive 11th root of unity.

(a) Find (with proof) the Galois group of F over $\mathbb{Q}$.

(b) Find (with proof) generators for each intermediate field between F and $\mathbb{Q}$.