Math 817–818 Qualifying Exam

Tuesday, May 29, 2007, 2–6 pm

Rules of the game:

(a) Solve *two* problems from each of the three parts, for a total of *six*.

(b) If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(c) **Justify all of your answers**.

(d) It is OK to use calculators, but only for routine arithmetic over the integers, rationals, reals, and complexes. Resist further temptations.

Section I: Groups

(1) Prove that there is no simple group of order $448 = 2^6 \cdot 7$.

(2) Let G be a group of order $1105 = 5 \cdot 13 \cdot 17$.

(a) Prove that G has a normal subgroup N of order $221 = 13 \cdot 17$.

(b) Prove that $|\mathrm{Aut}(N)| = 12 \cdot 16 = 192$.

(c) Prove that G is cyclic.

(3) Given an integer $n \geq 2$, let $G = \mathrm{SL}_n(\mathbb{R})$, the group of real $n \times n$ matrices with determinant 1. Let G act on $\mathbb{R}^n$ (column vectors) by left multiplication. Describe (with proof) the orbits of this action. (Think about the orbit of $[1\ 0\ \cdots\ 0]^t$.)

Section II: Linear Algebra

(4) (a) Let U and V be subspaces of a finite-dimensional vector space W. Prove that $\dim(U \cap V) \geq \dim(U) + \dim(V) - \dim(W)$.

(b) Let V_1, V_2, V_3 be subspaces of $\mathbb{R}^{10}$ with dimensions $6, 7$ and 8, respectively. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

(c) Find subspaces V_1, V_2, V_3 of $\mathbb{R}^{10}$ with dimensions $5, 7$ and 8, respectively, such that $V_1 \cap V_2 \cap V_3 = \{0\}$.

(5) Let A be the real 4×4 matrix each of whose entries is 1. Find an orthogonal matrix P and a diagonal matrix D such that $PAP^{\mathrm{t}} = D$. (By observing that A has rank 1 and that $[1\ 1\ 1\ 1]^{\mathrm{t}}$ is an eigenvector, you can avoid the drudgery of computing the characteristic polynomial.)

(6) Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 20 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 20 \\ 0 & 0 & 0 \end{bmatrix}$. Regard A and B as matrices over $\mathbb{Z}$.

(a) Express the cokernel of A (i.e., $\mathbb{Z}^3$/column space of A) in invariant factor form (i.e., in the form $\mathbb{Z}^r \oplus \mathbb{Z}/(d_1) \oplus \ldots \oplus \mathbb{Z}/(d_n)$, where $r \geq 0$, and for each i, one has $d_i \geq 2$ and $d_i \mid d_{i+1}$).

(b) Express the cokernel of A in elementary divisor form (that is, as a direct sum of cyclic groups, each of which is either infinite or of prime power order).

(c) Express the cokernel of B in elementary divisor form.

Section III: Rings and Fields

(7) Prove that $49x^5 - 7x^2 + 11x + 20$ is irreducible in $\mathbb{Q}[x]$. (Hint: Modulo 3 it factors as $(x^2+1)(x^3-x-1)$.)

(8) Let R be a commutative ring with 1. Use Zorn's lemma to prove that R has a minimal prime ideal. Recall that an ideal P is prime if $r \cdot s \in P$ implies $r \in P$ or $s \in P$. A prime ideal is said to be minimal when it does not properly contain any other prime ideal.

(9) Set $\zeta = e^{2\pi i/7}$, a primitive 7^{th} root of unity, and let σ be the automorphism of $\mathbb{Q}(\zeta)$ taking ζ to ζ^3.

(a) Prove that σ generates the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.

(b) Prove that $\mathbb{Q}(\zeta + \zeta^2 + \zeta^4)$ is the fixed field of σ^2.

(c) Find $[\mathbb{Q}(\zeta + \zeta^2 + \zeta^4) : \mathbb{Q}]$.

(10) Let $f \in \mathbb{Q}[x]$ be a polynomial of degree 4 whose Galois group is the full symmetric group S_4, and let $\alpha \in \mathbb{C}$ be a root of f. Prove that there are no fields strictly between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$.