Math 901–902 Qualify Exam

May 29, 2007, 2–6pm

Do two problems from each of the three sections, for a total of *six* problems.

If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

A. Groups and Character Theory

1. Consider the collection of groups G satisfying $|G| = 56 = 2^3 \cdot 7$ and there is a subgroup H of G that is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$.
   (a) Prove there are at least three such groups which are not isomorphic to each other.
   (b) Prove there are exactly two such groups (up to isomorphism) satisfying the additional condition that H is a normal subgroup of G.

2. For a group G, define subgroups $G^{(i)}$, for $i = 0, 1, \ldots$, recursively by $G^{(0)} = G$ and $G^{(i+1)} = (G^{(i)})'$ for $i \geq 0$. (Here, for a group H, H' is the derived subgroup of H, defined to be the subgroup generated by the set $\{xyx^{-1}y^{-1} \mid x, y \in H\}$.) Prove G is solvable if and only if $G^{(n)} = \{e\}$ for n sufficiently large.

3. Find, with justification, the complete character table for S_4, the permutation group on 4 letters. (There are many ways of doing this, but here is one tip that might help: Let $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4$ be a four-dimensional vector space over $\mathbb{C}$. Consider V as a $\mathbb{C}[S_4]$-module by defining $\sigma e_i := e_{\sigma(i)}$ for $\sigma \in S_4$ and $i \in \{1, 2, 3, 4\}$ and extending this action by linearity. Show that V decomposes as a $\mathbb{C}[S_4]$-submodule into the direct sum of two simple submodules, one of which gives rise to an irreducible degree 3 character for S_4.)

B. Field and Galois Theory

4. Let F be a field, $f(x) \in F[x]$ be a non-constant polynomial, E be a splitting field of $f(x)$ over F, and let $G = \mathrm{Aut}(E/F)$ (the group of field automorphisms of E that fix F element-wise). Prove G acts transitively on the set of roots of $f(x)$ in E if and only if $f(x)$ is irreducible. (Note: The field F does not necessarily have characteristic 0. Also, a group G acts transitively on a set X if for all $x, y \in X$ there exists $g \in G$ such that $gx = y$.)

5. Let E be a splitting field for $x^5 - 7$ over $\mathbb{Q}$ and $G = \mathrm{Gal}(E/\mathbb{Q})$.
   (a) Find intermediate fields K and L of $E/\mathbb{Q}$ (with $K \neq \mathbb{Q}$ and $L \neq \mathbb{Q}$) such that G is the semidirect product of $\mathrm{Gal}(E/K)$ and $\mathrm{Gal}(E/L)$.
   (b) Show there exist exactly five intermediate fields of $E/\mathbb{Q}$ which have degree 5 over $\mathbb{Q}$.

6. Let G be a finite cyclic group. Prove there exists a finite Galois extension of $\mathbb{Q}$ whose Galois group is isomorphic to G. (You may use without proof that for every integer m there exists a prime p such that $p \equiv 1 \pmod{m}$.)

C. Rings and Modules

7. Let R be a commutative ring and M and N finitely generated R-modules. Suppose M has finite length (i.e., has a composition series). Prove that $M \otimes_R N$ has finite length.

8. Let R be a finite-dimensional algebra over a field. Prove that R is a simple ring (i.e., a ring with no nontrivial two-sided ideals) if and only if R has a faithful simple left R-module.

9. Let R be a commutative ring.

   (a) Let M be an R-module. Prove that M is indecomposable if and only if $\mathrm{End}_R(M)$ has no nontrivial idempotents.

   (b) Let I be an ideal of R which contains a non-zero-divisor. Prove that $\mathrm{End}_R(I)$ is commutative.

10. Let R be a simple ring (i.e., a ring with no nontrivial two-sided ideals) which contains a left ideal which is simple as a left R-module. Prove that R is semisimple.