Math 817–818 Qualifying Exam

June 2008

Rules of the game:

(a) Solve *two* problems from each of the three parts, for a total of *six*.

For problems with multiple parts you can assume the results of earlier parts, even if you have not solved on them.

If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(b) **Justify all of your answers**.

(c) Each problem is worth 20 points. For problems with multiple parts, bold numbers in [**brackets**] indicate the number of points assigned for that part.

Section I: Groups

(1) Let G be a group and H and K subgroups. Recall that HK is the subset of G defined as $HK = \{hk \mid h \in H, k \in K\}$.

- (a) [**10**] Prove $HK = KH$ if and only if HK is a subgroup of G.
- (b) [**10**] Give an example (with justification) where HK is not a subgroup.

(2) Let p be a prime and G a group of order p^m for $m \geq 1$.

- (a) [**10**] Prove G contains a normal subgroup of order p.
- (b) [**10**] Prove every normal subgroup of G of order p is contained in the center.

(3) On simple groups.

- (a) [**5**] Prove that if G is a simple group and $\phi : G \to S_n$ is a homomorphism, then $\phi^{-1}(A_n)$ equals either G or the trivial group $\{e\}$.
- (b) [**15**] Prove that for all $m \geq 1$ there are no simple groups of order $2^m 7$.

Section II: Rings and Fields

(4) Prove $x^5 + 6x^3 + x^2 + 3x + 2$ is irreducible in $\mathbb{Q}[x]$.

(5) Let F be a finite field.

- (a) [**10**] Prove that the multiplicative group F^* of units of F is cyclic.
- (b) [**10**] Prove that the group of field automorphisms $\text{Aut}(F)$ of F is cyclic.

(6) Let $\omega \in \mathbb{C}$ be a primitive 16th root of unity and let $E = \mathbb{Q}(\omega)$.

(a) **[10]** Prove there are exactly three subfields F of E with $[F : \mathbb{Q}] = 4$.

(b) **[10]** For each field F in part (a), find (with justification) an element $\alpha \in F$ such that $F = \mathbb{Q}(\alpha)$.

Section III: Linear Algebra and Modules

(7) Let A be an $m \times n$ matrix with entries in a field F. Prove that $\operatorname{rank}(A) = r$ if and only if there exist invertible matrices P and Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

(where I_r is the $r \times r$ identify matrix and the 0's denote zero matrices of appropriate sizes).

(8) Consider 4×4 matrices A satisfying the equation $A^3 = I_4$.

(a) **[10]** Find the number of similarity classes of *real* matrices of this type, and give a representative of each class. Justify everything.

(b) **[10]** Find (with justification) the number of similarity classes of *complex* matrices of this type. You don't need to list representatives for these classes.

(9) Let R be a commutative ring and M a nonzero R-module. An R-submodule N of M is called *maximal* if $N \neq M$ and there are no proper R-submodules of M properly containing N.

(a) **[10]** Suppose M is finitely generated. Prove that there exists a maximal R-submodule of M.

(b) **[10]** Prove that if N is a maximal R-submodule of M, then M/N is isomorphic as an R-module to R/m, where m is a maximal ideal of R.