Math 817–818 Qualifying Exam

June 2009

Instructions:

(a) Solve *two* problems from each of the three parts, for a total of *six*.

For problems with multiple parts you can assume the results of earlier parts, even if you have not solved them.

If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(b) **Justify all of your answers**.

(c) Each problem is worth 20 points. For problems with multiple parts, bold numbers in [**brackets**] indicate the number of points assigned for that part.

Section I: Groups

(1) Let p be a prime number, G be a finite p-group, Z the center of G, and $N \neq \{1\}$ a normal subgroup of G.

(a) [**12 pts**] Prove that $N \cap Z \neq \{1\}$.

(b) [**8 pts**] Prove that N contains a subgroup H such that H is normal in G and $p = [N : H]$. (Hint: Use part (a) and induction.)

(2) Let G be a group and K a finite cyclic normal subgroup of G. Prove that $G' \subseteq C_G(K)$, where G' is the commutator subgroup of G and $C_G(K) = \{g \in G \mid gk = kg \text{ for all } k \in K\}$. (Hint: Consider an appropriate action of G on K.)

(3) Let G be a group of order $5 \cdot 11 \cdot 13^2$. Suppose G contains an element of order 55. Prove that G is abelian.

Section II: Rings and Fields

(4) Let R be a commutative ring with identity and $r \in R$ such that r is not nilpotent. Let Λ be the set of all ideals I of R such that $r^n \notin I$ for all $n \geq 1$.

(a) [**10 pts**] Prove that Λ has a maximal element.

(b) [**10 pts**] Prove that any maximal element of Λ is a prime ideal.

(5) Let E/F be an algebraic extension. Suppose $f : E \to E$ is a field homomorphism which fixes F. Prove that f is an automorphism of E.

(6) Let E be the field $\mathbb{Q}(\sqrt[3]{5}, \sqrt{-3})$.

(a) [**6 pts**] Show that E is a splitting field for $x^3 - 5$ over $\mathbb{Q}$.

(b) **[8 pts]** Find the Galois group of $E/\mathbb{Q}$. That is, explicitly describe all automorphisms of E and identify the group structure (e.g., by showing it is isomorphic to some well-known group).

(c) **[6 pts]** Find a primitive element of E over $\mathbb{Q}$.

Section III: Linear Algebra and Modules

(7) Prove one (and only one) of the following statements. Solving either problem is worth 20 points.

I. Let A be a square matrix with entries in an arbitrary field. Prove that A is similar to its transpose.

II. Let R be a Euclidean domain, A an $m \times n$ matrix with elements from R, and A^{T} the transpose matrix of A. Recall that $\operatorname{Coker}(A)$ denotes the quotient of R^m by the submodule generated by the columns of A.

(a) **[10 pts]** Prove that the torsion submodules of $\operatorname{Coker}(A)$ and $\operatorname{Coker}(A^{\mathsf{T}})$ are isomorphic.

(b) **[10 pts]** Prove that the modules $\operatorname{Coker}(A)$ and $\operatorname{Coker}(A^{\mathsf{T}})$ are isomorphic if and only if $m = n$.

(8) Let R be a domain and M an R-module. Recall that a subset S of M is called a *maximal linearly independent set* of M if S is linearly independent and any subset of M properly containing S is linearly dependent.

(a) **[10 pts]** Let T be a linearly independent subset of M. Prove that T is contained in some maximal linearly independent subset of M.

(b) **[10 pts]** Let T be a linearly independent subset of M and N the R-submodule of M generated by T. Prove that T is a maximal linearly independent subset if and only if M/N is torsion.

(9) Consider the matrix A below with rational coefficients.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

(a) **[5 pts]** Find the rank of A.

(b) **[5 pts]** Find the characteristic polynomial of A.

(c) **[5 pts]**Find the eigenvalues of A.

(d) **[5 pts]**Find the Jordan canonical form of A.