# 817-818 Qualifying Exam — June 4, 2010

*Instructions:* Solve two problems from each section, for a total of six problems. For problems with multiple parts you can assume the results of earlier parts, even if you have not solved them. If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial. Justify all of your answers.

**Section I. Groups**

1. Recall that the *centralizer* of a subgroup $H$ in a group $G$ is
$$C_G(H) = \{g \in G \,|\, gh = hg \text{ for all } h \in H\}.$$
   (a) Prove that if $H$ is normal in $G$, then $C_G(H)$ is normal in $G$.

   (b) Prove that if $H$ is normal in $G$, then $G/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$ (the group of automorphisms of $H$).

2. (a) Suppose $H$ is a subgroup of a group $G$ and $[G : H] = 7$. Prove $G$ contains a normal subgroup $N$ such that $N \subset H$ and $[G : N] \leq 7!$.

   (b) Prove $7!$ is the best possible bound for the previous part — i.e., prove there is a group $G$ and a subgroup $H$ with $[G : H] = 7$ such that for every normal subgroup $N$ of $G$ with $N \subset H$, we have $[G : N] \geq 7!$.

3. Suppose $G$ is a simple group of order $168 = 2^3 \cdot 3 \cdot 7$. (Yes, there is such a group.)

   (a) How many elements of order 7 does $G$ have?

   (b) Show that $G$ has at least 14 elements of order 3.

**Section II. Linear Algebra and Modules**

4. Let $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$, with entries in $\mathbb{C}$.

   (a) Find the Jordan canonical form of $A$.

   (b) Let $B = \begin{bmatrix} 0 & -9 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 1 & 6 \end{bmatrix}$, with entries in $\mathbb{C}$. Is $A$ similar to $B$?

5. Let $R$ be a ring with identity and let $M$ be a left $R$-module. Recall that the *annihilator* of $M$ in $R$ is
$$\text{ann}_R(M) = \{r \in R \,|\, rm = 0 \text{ for all } m \in M\}.$$
   (a) Prove that $\text{ann}_R(M)$ is a 2-sided ideal of $R$.

   (b) Suppose $M$ is an abelian group (i.e., a $\mathbb{Z}$-module) such that $|M| = 400$ and $\text{ann}_{\mathbb{Z}}(M)$ is the ideal generated by 20. How many possibilities, up to isomorphism, are there for $M$?

6. Let $F$ be a field and $V$ a vector space (not necessarily finite-dimensional) over $F$. Prove that every linearly independent subset of $V$ is contained in a basis for $V$.

**Section III. Rings, Fields and Galois Theory**

7. Let $R$ be an integral domain with field of fractions $Q$. Let $P$ be a prime ideal of $R$ and let

$$S = \left\{ \frac{r}{d} \in Q \,|\, d \notin P \right\}.$$

   (a) Show that $S$ is a subring of $Q$.

   (b) Show that

$$I = \left\{ \frac{p}{d} \,|\, p \in P, d \notin P \right\}$$

   is a prime ideal of $S$.

8. Prove $\mathbb{Z}[2i] = \{a + 2bi \,|\, a \text{ and } b \text{ are integers}\}$ is not a PID. *Hint*: One method is to use (with proof) the fact that $2 + 2i$ is irreducible in this ring.

9. Consider $f(x) = x^6 + 3 \in \mathbb{Q}[x]$.

   (a) Let $\alpha$ be a root of $f(x)$ and prove $\mathbb{Q}(\alpha)$ is Galois over $\mathbb{Q}$. (Hint: First show $\frac{\alpha^3+1}{2}$ is a primitive 6-th root of unity.)

   (b) Find the Galois group of $\mathbb{Q}(\alpha)/\mathbb{Q}$.