Math 817–818 Qualifying Exam

June 2012

Rules of the game:

(a) Solve *two* problems from each of the three parts, for a total of *six*.

For problems with multiple parts you can assume the results of earlier parts, even if you have not solved them.

If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(b) **Justify all of your answers**.

(c) Bold numbers in [**brackets**] indicate the number of points assigned for a complete solution.

Section I: Groups

(1) Let G be a group of order $231(= 3 \cdot 7 \cdot 11)$.

(a) [**10**] Prove that G has a unique 11-Sylow subgroup.

(b) [**10**] Prove that the 11-Sylow subgroup is contained in the center of G.

(2) Let G be a group with a subgroup H so that $[G : H] = n < \infty$.

(a) [**10**] Prove that there is a normal subgroup of G, N, so that $N \subseteq H$ and $[G : N] \leq n!$.

(b) [**10**] Prove that if G is finitely generated, there are most finitely many subgroups with index n. (Hint: you might want to consider maps $G \to S_n$.)

(3) [**20**] Let G be a finite p-group and Z its center. If $N \neq \{e\}$ is a normal subgroup of G, prove that $N \cap Z \neq \{e\}$.

Section II: Linear Algebra and Modules

(4) Let V be a subspace of a finite-dimensional vector space, W. Recall that a subspace U of W is called a *complement* of V if $U \oplus V = W$.

Prove the following statements.

(a) [**7**] Every complement of V has dimension $\dim W - \dim V$.

(b) [**7**] If V is not 0 or W, then V has more than one complement.

(c) [**6**] If T is a subspace of W with $\dim T + \dim V > \dim W$, then $T \cap V$ is non-zero.

(5) Let R be a commutative integral domain and M an R-module. Recall that a subset S of M is called a *maximal linearly independent* set of M if S is linearly independent and any subset of M properly containing S is linearly dependent.

(a) [**10**] Let T be a linearly independent subset of M. Prove that T is contained in some maximal linearly independent subset of M.

(b) [**10**] Let T be a linearly independent subset of M and N the R- submodule of M generated by T. Prove that T is a maximal linearly independent subset if and only if M/N is torsion.

(6) Let V be the set of all $r \times s$ matrices over $\mathbb{R}$, let G denote the group $GL_r(\mathbb{R}) \times GL_s(\mathbb{R})$, and set

$$(A, B) \cdot M = AMB^{-1} \quad \text{for all} \quad M \in V \quad \text{and} \quad (A, B) \in G.$$

(a) **[5]** Prove that the formula above defines a group action.

(b) **[10]** Prove that each orbit contains a matrix $M = (m_{ij})$ such that

$$m_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 0 \text{ or } 1 & \text{when } i = j. \end{cases}$$

(c) **[5]** How many orbits are there?

Section III: Rings and Fields

(7) **[20]** Let R be a commutative integral domain and K its field of fractions. Let a and b be nonzero elements of R, such that $(a) \cap (b) = (ab)$. Let $f : R[x] \to K$ be the unique ring homomorphism with $f(r) = r/1$ for $r \in R$ and $f(x) = b/a$. Prove that a polynomial $p(x) \in R[x]$ satisfies $f(p(x)) = 0$ if and only if $p(x) = (ax - b)q(x)$ for some polynomial $q(x) \in R[x]$.
(Hint: one way is to use induction on $\deg(p(x))$.)

(8) Let a be an integer that is not a square, and set

$$\mathbb{Z}[\sqrt{a}] = \{m + n\sqrt{a} \in \mathbb{C} \mid m, n \in \mathbb{Z}\}.$$

(a) **[5]** Prove that $\mathbb{Z}[\sqrt{a}]$ is a subring of $\mathbb{C}$.

When b is an integer that is not a square, prove the following assertions:

(b) **[5]** There is an isomorphism of *abelian groups* (under addition) $\mathbb{Z}[\sqrt{a}] \cong \mathbb{Z}[\sqrt{b}]$.

(c) **[10]** There is an isomorphism of *rings* $\mathbb{Z}[\sqrt{a}] \cong \mathbb{Z}[\sqrt{b}]$ if and only if $a = b$.

(9) Suppose A and B are subfields of a field extension K/F with $[A : F]$ and $[B : F]$ both finite. Let E be the subfield of K generated by A and B.

(a) **[7]** Show that $[E : F] \leq [A : F][B : F]$.

(b) **[8]** Prove that equality holds when $[A : F]$ and $[B : F]$ are relatively prime

(c) **[5]** Prove there are two subfields of $\mathbb{R}/\mathbb{Q}$, A and B, neither contained in the other, so that the inequality in part (a) is strict.