Math 817–818 Qualifying Exam

June 2014

Instructions:

(a) Solve *two* problems from each of the three parts, for a total of *six*.

The problems have multiple parts; usually each part is a step towards the next. Assume the results of earlier parts, even if you do not solve them.

If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(b) **Justify all of your answers**.

(c) Bold numbers in [**brackets**] indicate the number of points assigned for a complete solution.

Section I: Group Theory

(1) Let G be a finite group, p a prime dividing the order of G, and H a normal subgroup of order p^m for some $m > 0$. Prove the following statements.

(a) [**8**] H is contained in every Sylow p-subgroup of G.

(b) [**12**] H contains a nontrivial abelian subgroup that is normal in G.

(2) Let G be a group of order p^2q where p and q are distinct primes.

(a) [**10**] Prove that G contains a normal Sylow subgroup.

(b) [**10**] Suppose $p < q$ and the Sylow p-subgroup is cyclic and normal. Prove that G is abelian.

(3) Let G be a group that acts on a set A, and H a subgroup of G such that for any $a, b \in A$ there exists a unique $h \in H$ with $ha = b$.

(a) [**10**] Prove that for every $a \in A$, $G = HG_a$ and $H \cap G_a = \{1\}$, where $G_a = \{g \in G \mid ga = a\}$.

(b) [**10**] Prove that if $H \subseteq Z(G)$ then for every $a \in A$, G is the internal direct product of H and G_a.

Section II: Field Theory and Galois theory

(4) Let $F \subseteq K \subseteq L$ be extensions of fields, not necessarily finite.

(a) [**14**] Prove that K/F and L/K are algebraic if and only if L/F is algebraic.

(b) [**6**] Give an example where K/F and L/K are Galois but L/F is not Galois.

(5) Let ζ denote a primitive 17th root of unity, over $\mathbb{Q}$.

(a) **[5]** Prove that $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension, of degree 16.

(b) **[5]** Describe an explicit isomorphism $\mathbb{Z}/16 \xrightarrow{\cong} \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

(c) **[10]** Describe a primitive generator for the fixed subfield of the subgroup $\mathbb{Z}/4$ of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

(6) Let K/F be a finite Galois field extension, $G = \mathrm{Gal}(K/F)$, and $n = |G|$. Let α be an element of K and $m(x)$ its minimal polynomial over F; set $d = \deg m(x)$.

(a) **[10]** Prove there are d distinct elements in the set $\{\sigma(\alpha) \mid \sigma \in G\}$.

(b) **[10]** Prove

$$\prod_{\sigma \in G} (x - \sigma(\alpha)) = m(x)^{n/d}.$$

Section III: Ring theory and Linear Algebra

(7) Let V be a finite dimensional vector space over a field F and $f\colon V \to V$ an F-linear transformation. Prove the following assertions.

(a) **[7]** There exists an integer $s \geq 0$ such that for $n \geq s$ there are equalities

$$\ker(f^n) = \ker(f^s) \quad \text{and} \quad \mathrm{image}(f^n) = \mathrm{image}(f^s)$$

(b) **[7]** $\ker(f^s) \cap \mathrm{image}(f^s) = \{0\}$ for any s as above.

(c) **[6]** $V = \ker(f^s) \oplus \mathrm{image}(f^s)$ for any s as above.

(8) The *trace* of a square matrix A, denoted $\mathrm{Tr}(A)$, is the sum of its diagonal entries. Prove the following assertions.

(a) **[8]** $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any $n \times n$ matrices A, B.

(b) **[12]** Use (a) to prove that the trace of a matrix A over $\mathbb{C}$ is the sum of its eigenvalues (with multiplicities).

(9) Let R be a commutative ring, and set $I = \{r \in R \mid r^n = 0 \text{ for some integer } n\}$. Prove that following assertions.

(a) **[10]** I is an ideal in R.

(b) **[10]** If R/I is a field, then each element of R is either a unit or in I.