Math 817–818 Qualifying Exam

June 2016

Instructions:

(a) Solve *two* problems from each of the three parts, for a total of *six*.

Each problem will be graded out of 20 points. Some problems have multiple parts, in which case the point values are given. You may assume the results of earlier parts, even if you do not solve them.

If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(b) Justify all of your answers.

Section I: Group Theory

Do *two* of the following three problems.

1. Let G be a (not necessarily finite) group and $K \leq G$ a subgroup of index $n < \infty$. Define

$$N = \bigcap_{g \in G} gKg^{-1}$$

(i.e., N is the interesection of all the conjugates of K).

(a) (10 points) Prove N is a the largest normal subgroup of G that is contained in K.

(b) (10 points) Prove $[G : N]$ divides $n!$.

2. Prove that if G is a finite group of odd order, then for any non-identity element $x \in G$, x is not conjugate to x^{-1}.

3. Groups of order 75.

(a) (10 points) Prove that if G is a group of order 75 that contains an element of order 25, then G is cyclic.

(b) (10 points) Prove there exists a non-abelian group of order 75.

Section II: Field Theory and Galois theory

Do *two* of the following three problems.

4. Let L be the splitting field over $\mathbb{Q}$ of the polynomial $x^3 - 7$.

(a) (10 points) Find all intermediate fields E with $\mathbb{Q} \subseteq E \subseteq L$ (including possibly L and $\mathbb{Q}$) such that E is Galois over $\mathbb{Q}$.

(b) (10 points) For each field E you found in (a), find with justification a primitive generator (i.e., find $\alpha \in E$ so that $E = \mathbb{Q}(\alpha)$).

5. Let p be any positive prime integer.

(a) (5 points) Prove that if $p = k^2 + 1$ for some integer k, then p is not an irreducible element of $\mathbb{Z}[i]$.

(b) (15 points) Prove $x^4 - p$ is irreducible in $\mathbb{Q}(i)[x]$.

6. For fields E and F, we say E is a *finite splitting field over* F if E is the splitting field over F of some polynomial $f(x) \in F[x]$.

Assume E and K are both finite splitting fields over $\mathbb{Q}$ and prove $E \cap K$ is also a finite splitting field over $\mathbb{Q}$.

Section III: Ring theory, Module theory and Linear Algebra

Do *two* of the following three problems.

7. Let F be a field and recall that a square matrix A with entries in F is called *nilpotent* if $A^m = 0$ for some positive integer m.

   (a) (7 points) Prove that if A is an $n \times n$ nilpotent matrix, then $A^n = 0$.

   (b) (7 points) Assume $n \leq 3$ and prove that two $n \times n$ nilpotent matrices are similar if and only if they have the same rank. (Recall the rank of a matrix is the dimension of the vector space spanned by its columns.)

   (c) (6 points) Give an example, with justification, of two 4×4 nilpotent matrices that have the same rank but are not similar.

8. Let R be a commutative ring and x an indeterminant. Prove that $R[x]$ is a principal ideal domain (PID) if and only if R is a field.

9. Find, with justification, a complete and non-redundant list of conjugacy class representatives for the group $\mathrm{GL}_2(\mathbb{F}_3)$, where $\mathbb{F}_3$ is the field with three elements.