Math 817–818 Qualifying Exam

January 2016

Instructions:

- Solve *two* problems from each of the three parts, for a total of *six*.
- Some problems have multiple parts, in which case the point values are given. You may assume the results of earlier parts, even if you do not solve them.
- If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.
- Justify all of your answers.

Section I: Group Theory

Do *two* of the following three problems.

1. Let p be a prime and H a finite p-group.

   (a) (10 points) Suppose H acts on the finite set S, and let S_0 be the fixed points of the action, i.e.,

   $$S_0 = \{x \in S \mid hx = x \text{ for all } h \in H\}.$$

   Prove that $|S| \equiv |S_0| \pmod{p}$.

   (b) (10 points) Suppose G is a finite group and $H \leq G$ (where H is still a finite p-group). Prove that $[N_G(H) : H] \equiv [G : H] \pmod{p}$.

2. A normal subgroup H of G is called a *direct factor* of G if there is some normal subgroup K of G such that $G = H \times K$ (internal direct product).

   (a) (10 points) If H is a direct factor of G, L is any group, and $\phi : H \to L$ is any group homomorphism, prove that ϕ extends to a homomorphism $\tilde{\phi} : G \to L$ such that $\tilde{\phi}(h) = \phi(h)$ for all $h \in H$.

   (b) (10 points) Provide a counter-example (with justification) to the previous part if H is assumed to be a normal subgroup of G, but not a direct factor. *Hint*: One such example involves $G = \mathbb{Z}$.

3. Suppose G is a group of order $5 \cdot 7 \cdot 23^2$ and that G contains an element of order 35. Prove G is abelian.

Section II: Field Theory and Galois theory

Do *two* of the following three problems.

4. Let E be the splitting field of $x^4 + 7 \in \mathbb{Q}[x]$.

   (a) (10 points) Prove that $\sqrt[4]{28} \in E$.

   (b) (10 points) Find a basis for E as a $\mathbb{Q}$-vector space.

5. Let F be a field and G a finite subgroup of F^*, the multiplicative group of units of F. Prove that G is cyclic.

6. Let L be a finite Galois field extension of $\mathbb{Q}$. Let E and F be subfields of L such that $EF = L$, $E/\mathbb{Q}$ is normal, and $E \cap F = \mathbb{Q}$. Prove that $[L : \mathbb{Q}] = [E : \mathbb{Q}][F : \mathbb{Q}]$.

Section III: Ring theory, Module theory and Linear Algebra

Do *two* of the following three problems.

7. Let V be a finite dimensional vector space over a field F and let $\theta : V \to V$ be an F-linear operator on V. Prove θ is diagonalizable over F if and only if its minimum polynomial $p(x)$ factors into distinct linear terms in $F[x]$ (i.e., $p(x) = \prod_{i=1}^{d}(x - a_i)$ for distinct elements $a_1, \ldots, a_n$ of F).

8. Prove the ideal $I = (2, 1 + \sqrt{-13})$ of the commutative ring $R = \mathbb{Z}[\sqrt{-13}]$ is not a principal ideal.

9. Let I be an ideal in a commutative ring R, let M and N be R-modules and let $f : M \to N$ be an R-module homomorphism.

   (a) (8 points) Prove there is a unique R-module homomorphism $\overline{f} : M/IM \to N/IN$ such that $\overline{f} \circ p = q \circ f$, where $p : M \twoheadrightarrow M/IM$ and $q : N \twoheadrightarrow N/IN$ are the canonical quotient maps.

   (b) (12 points) Prove that if $I^2 = 0$ and $\overline{f}$ is surjective, then so is f. (Recall that I^2 is the ideal generated by all elements of the form ab, where $a, b \in I$.)